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Student Edition

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LONG + LIVE + MATH
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Mathematics is so much more than memorizing rules. It is learning to reason, to make connections, and to make sense of the world. We believe in Learning by Doing—\textsuperscript{TM} you need to actively engage with the content if you are to benefit from it. The lessons were designed to take you from your intuitive understanding of the world and build on your prior experiences to then learn new concepts. My hope is that these instructional materials help you build a deep understanding of math.

Sandy Bartle Finocchi, Senior Academic Officer

My hope is that as you work through this course, you feel capable—capable of exploring new ideas that build upon what you already know, capable of struggling through challenging problems, capable of thinking creatively about how to fix mistakes, and capable of thinking like a mathematician.

Amy Jones Lewis, Director of Instructional Design

At Carnegie Learning we have created an organization whose mission and culture is defined by your success. Our passion is creating products that make sense of the world of mathematics and ignite a passion in you. Our hope is that you will enjoy our resources as much as we enjoyed creating them.

Barry Malkin, CEO

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1. Learning Goals
Learning goals are stated for each lesson to help you take ownership of the learning objectives.

2. Connection
Each lesson begins with a statement connecting what you have learned with a question to ponder.

Return to this question at the end of this lesson to gauge your understanding.

---

WARM UP
Write a fraction to represent each situation.
1. the number of boys in your math class compared to the number of students in the class
2. the number of girls in your math class compared to the number of students in the class
3. the number of students in your math class that are absent today compared to the total number of students in the class
4. the number of students in your math class that are in attendance today compared to the total number of students in your class

In elementary school, you made many comparisons using addition and subtraction. You answered questions like, “If Johnny has 9 apples and Suzie has 12 apples, who has more apples?” Is there another way to compare values?

---

LEARNING GOALS
• Distinguish between additive and multiplicative relationships between two quantities.
• Understand the concept of a ratio: a ratio represents a multiplicative comparison between two quantities.
• Write ratios in different forms and use ratio language to represent relationships between two quantities.
• Distinguish between part-to-part and part-to-whole ratios.
• Understand that fractions are part-to-whole ratios between two quantities.
• Understand that percents are part-to-whole ratios between a quantity and 100.

KEY TERMS
• additive reasoning
• multiplicative reasoning
• ratio
• percent
Predict the Score

The Crusaders and the Blue Jays just finished the first half of their basketball game.

<table>
<thead>
<tr>
<th></th>
<th>Halftime Score</th>
<th>Final Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crusaders</td>
<td>30</td>
<td>?</td>
</tr>
<tr>
<td>Blue Jays</td>
<td>20</td>
<td>?</td>
</tr>
</tbody>
</table>

1. Predict the final score. Explain your reasoning.
4. Activities

You are going to build a deep understanding of mathematics through a variety of activities in an environment where collaboration and conversations are important and expected.

You will learn how to solve new problems, but you will also learn why those strategies work and how they are connected to other strategies you already know.

Remember:

• It’s not just about answer-getting. The process is important.
• Making mistakes is a critical part of learning, so take risks.
• There is often more than one way to solve a problem.

Activities may include real-world problems, sorting activities, worked examples, or analyzing sample student work.

Be prepared to share your solutions and methods with your classmates.
5. Talk the Talk
Talk the Talk gives you an opportunity to reflect on the main ideas of the lesson.

- Be honest with yourself.
- Ask questions to clarify anything you don’t understand.
- Show what you know!

Don’t forget to revisit the question posed on the lesson opening page to gauge your understanding.

---

C01_SE_FM.indd   FM-13
1/12/19   8:13 PM
6. Write
Reflect on your work and clarify your thinking.

7. Remember
Take note of the key concepts from the lesson.

8. Practice
Use the concepts learned in the lesson to solve problems.

9. Stretch
Ready for a challenge?

10. Review
Remember what you’ve learned by practicing concepts from previous lessons and topics.
Problem Types You Will See

**WORKED EXAMPLE**

Determine the quantity in pounds that is equivalent to 4.5 kilograms.

**Scaling Up**

\[
\frac{1 \text{ kg}}{2.2 \text{ lb}} \times 4.5 = \frac{4.5 \text{ kg}}{9.9 \text{ lb}}
\]

**Unit Analysis**

\[
\frac{4.5 \text{ kg}}{1} \times \frac{2.2 \text{ lb}}{1 \text{ kg}} = 9.9 \text{ lb}
\]

\[
\frac{1 \text{ kg}}{2.2 \text{ lb}} = \frac{4.5 \text{ kg}}{9.9 \text{ lb}}
\]

\[
4.5 \text{ kg} = 9.9 \text{ lb}
\]

**Thumbs Up**

When you see a Thumbs Up icon:

- Take your time to read through the correct solution.
- Think about the connections between steps.

**Ask Yourself:**

- Why is this method correct?
- Have I used this method before?

**Thumbs Down**

When you see a Thumbs Down icon:

- Take your time to read through the incorrect solution.
- Think about what error was made.

**Ask Yourself:**

- Where is the error?
- Why is it an error?
- How can I correct it?

Christopher and Max want to determine the number of miles in 31,680 feet using unit analysis.

**Max**

\[
\frac{31,680 \text{ ft}}{5280 \text{ ft}} = 6 \text{ mi}
\]

**Christopher**

\[
31,680 \text{ ft} \times \frac{5280 \text{ ft}}{1 \text{ mi}} = 167,270,400 \text{ mi}
\]
Tim and Dan love cereal, but don’t want to spend a lot of money. After scanning the aisle in the grocery store for the lowest prices, the boys make the following statements.

- Tim says, “I found Sweetie Oat Puffs for $0.14 per ounce. That’s the cheapest cereal in the aisle!”
- Dan replies, “It’s not cheaper than Sugar Hoops! The unit price for that is 6.25 oz per dollar.”

Who is correct? Explain your reasoning.
The Crew is here to help you on your journey. Sometimes they will remind you about things you already learned. Sometimes they will ask you questions to help you think about different strategies. Sometimes they will share fun facts. They are members of your group—someone you can rely on!

Teacher aides will guide you along your journey. They will help you make connections and remind you to think about the details.
With practice, you can develop the habits of mind of a productive mathematical thinker.

**Mathematical Practices**

The types of activities within this book require you to make sense of mathematics and to demonstrate your reasoning through problem solving, writing, discussing, and presenting. Effective communication and collaboration are essential skills of a successful learner.

Each activity is denoted with an icon that represents a practice or pair of practices intentionally being developed. To help develop these habits of mind ask yourself the types of questions listed as you work.

**Make sense of problems and persevere in solving them.**

Questions to ask:
- What is this problem asking and what is my plan for answering it?
- What tools do I need to solve this problem?
- Does my answer make sense?

**Reason abstractly and quantitatively.**

**Construct viable arguments and critique the reasoning of others.**

Questions to ask:
- What representation can I use to solve this problem?
- How can this problem be represented with symbols and numbers?
- How can I explain my thinking?
- How does my strategy compare to my partner’s?

I hope that every once in a while you will see something that you weren’t quite expecting. These are my favorite parts! Because I <3 being confused at first, and then figuring it out.

Josh Fisher, Instructional Designer
### Model with mathematics.  
- Use appropriate tools strategically.

Questions to ask:
- What expression or equation could represent this situation?
- What tools would help me solve this problem?
- What representations best show my thinking?
- How does this answer make sense in the context of the original problem?

### Attend to precision.

Questions to ask:
- Is my answer accurate?
- Did I use the correct units or labels?
- Is there a more efficient way to solve this problem?
- Is there more sophisticated vocabulary that I could use in my explanation?

### Look for and make use of structure.  
- Look for and express regularity in repeated reasoning.

Questions to ask:
- What characteristics of this expression or equation are made clear through this representation?
- How can I use what I know to explain why this works?
- Can I develop a more efficient method?
- How could this problem help me to solve another problem?

"This book is your place to record your thoughts, your conjectures, your mistakes, your strategies, and your ‘ah-has’ about the mathematics you need to learn this year. Don’t erase when you make mistakes; cross it out so that you can still see your original thinking. Learn from your mistakes and grow your brain.

Kelly Edenfield, Instructional Designer"
Academic Glossary

There are important terms you will encounter throughout this book. It is important that you have an understanding of these words as you get started on your journey through the mathematical concepts. Knowing what is meant by these terms and using these terms will help you think, reason, and communicate your ideas.

**Related Phrases**
- Examine
- Evaluate
- Determine
- Observe
- Consider
- Investigate
- What do you notice?
- What do you think?
- Sort and match

**ANALYZE**

**Definition**
To study or look closely for patterns. Analyzing can involve examining or breaking a concept down into smaller parts to gain a better understanding of it.

**Ask Yourself**
- Do I see any patterns?
- Have I seen something like this before?
- What happens if the shape, representation, or numbers change?

**Related Phrases**
- Show your work
- Explain your calculation
- Justify
- Why or why not?

**EXPLAIN YOUR REASONING**

**Definition**
To give details or describe how to determine an answer or solution. Explaining your reasoning helps justify conclusions.

**Ask Yourself**
- How should I organize my thoughts?
- Is my explanation logical?
- Does my reasoning make sense?
- How can I justify my answer to others?
**REPRESENT**

**Definition**
To display information in various ways. Representing mathematics can be done using words, tables, graphs, or symbols.

**Ask Yourself**
- How should I organize my thoughts?
- How do I use this model to show a concept or idea?
- What does this representation tell me?
- Is my representation accurate?

**ESTIMATE**

**Definition**
To make an educated guess based on the analysis of given data. Estimating first helps inform reasoning.

**Ask Yourself**
- Does my reasoning make sense?
- Is my solution close to my estimation?

**DESCRIBE**

**Definition**
To represent or give an account of in words. Describing communicates mathematical ideas to others.

**Ask Yourself**
- How should I organize my thoughts?
- Is my explanation logical?
- Did I consider the context of the situation?
- Does my reasoning make sense?
The lessons in this module build on what you already know about area, number properties, and volume. You will learn to approach a problem by decomposing (taking apart) or composing (putting together) objects and numbers. You will examine the relationships between numbers and shapes, using area models to solve problems. You will strengthen your skills with fraction operations and use decimal operations to solve volume and surface area problems.

**Topic 1  Factors and Area** .................................................. M1-3
**Topic 2  Positive Rational Numbers** ................................. M1-67
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TOPIC 1
Factors and Area

Architects, city planners, landscape gardeners, and others determine areas using measuring and multiplication.

Lesson 1
Taking Apart Numbers and Shapes
Writing Equivalent Expressions Using the Distributive Property .......................... M1-7

Lesson 2
All About That Base...and Height
Area of Triangles and Quadrilaterals ................................................................. M1-15

Lesson 3
Slicing and Dicing
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Lesson 4
Searching for Common Ground
Common Factors and Common Multiples ......................................................... M1-39

Lesson 5
Composing and Decomposing Numbers
Least Common Multiple and Greatest Common Factor ....................................... M1-51
Module 1: Composing and Decomposing

TOPIC 1: FACTORS AND AREA
This topic integrates numeric concepts like the Distributive Property with the geometric concept of area. Students use their flexibility with decomposing shapes to decompose numbers into factors and to apply the Distributive Property to compute products efficiently. From their knowledge of rectangles and area, students also develop the formula for the area of parallelograms, triangles, trapezoids, and composite figures. Finally, students use their knowledge of factors to determine greatest common factors and least common multiples of numbers.

Where have we been?
In previous years, students have learned about area and number properties as well as simple closed shapes such as parallelograms and triangles. All of these concepts are used in this topic to connect composing and decomposing shapes to composing and decomposing numbers.

Where are we going?
This topic sets the stage for seeing structure in numbers and shapes. Recognizing different structures will be important when dealing with fractions, decimals, percents, and algebraic expressions and equations later in the course.

Using Area Models to Represent Products and Factors
An area model shows the product of two factors, such as $5 \times 27$. The model can be decomposed to show that the product of the numbers can also be decomposed.

\[5 \times 27 = 135\]

\[5 \times 27 = 5 \times (20 + 7) = (5 \times 20) + (5 \times 7) = 100 + 35 = 135\]
**Myth: “I don’t have the math gene.”**

Let’s be clear about something. There isn’t a gene that controls the development of mathematical thinking. Instead, there are probably hundreds of genes that contribute to it. A recent study suggests that mathematical thinking arises from the ability to learn a language. Given the right input from the environment, children learn to speak without formal instruction. They can learn number sense and pattern recognition the same way.

To further nurture your child’s mathematical growth, attend to the learning environment. You can think of it as providing a nutritious mathematical diet that includes: discussing math in the real world, offering encouragement, being available to answer questions, allowing your student to struggle with difficult concepts, and providing space for plenty of practice.

#mathmythbusted

**Talking Points**

You can further support your student’s learning by asking questions about the work they do in class or at home. Your student is learning to think flexibly about multiplication, area, and number properties.

**Questions to Ask**

- How does this problem look like something you did in class?
- Can you show me the strategy you used to solve this problem? Do you know another way to solve it?
- Does your answer make sense? Why?
- Is there anything you don’t understand? How can you use today’s lesson to help?

**Some Things to Look For**

Look for real-life examples of shapes that are composed of two or more different shapes. How can you estimate the entire area?

**Key Terms**

**numeric expression**

A numeric expression is a mathematical phrase containing numbers and operations.

**Distributive Property**

The Distributive Property states that for any numbers $a$, $b$, and $c$, $a(b + c) = ab + ac$.

**altitude**

The altitude of a parallelogram is the perpendicular distance from the base of the parallelogram to the opposite side, represented by a line segment.

**greatest common factor (GCF)**

The GCF is the largest factor two or more numbers have in common.

**least common multiple (LCM)**

The LCM is the smallest multiple (other than zero) that two or more numbers have in common.
LEARNING GOALS
• Write, read, and evaluate equivalent numeric expressions.
• Identify the adjacent side lengths of a rectangle as factors of the area value.
• Identify parts of an expression, such as the product and the factors.
• Write equivalent numeric expressions for the area of a rectangle by decomposing one side length into the sum of two numbers.
• Apply the properties of operations, such as the Distributive Property, to rewrite the product of two factors.

KEY TERMS
• numeric expression
• equation
• Distributive Property

You have learned how to operate with numbers using different strategies. Sometimes taking apart numbers before you operate can highlight important information or make calculations easier. How can you use these strategies to express number sentences in different ways?
Form of 24

Consider the number 24. What comes to mind?

1. Write five different numeric expressions for the number 24.

2. Share your numeric expressions with your classmates.
   a. Did you and your classmates use common strategies to write your expressions? Explain.
   b. How many possible numeric expressions could you write for this number?

3. What do you notice about the collected set of numeric expressions representing 24?
Consider the equation $5 \times 27 = 135$.

An area model to represent the product of 5 and 27 is shown. The area is 135 square units.

Let’s think about other ways to represent this area.

1. Draw a line to split one side length of the area model into two parts to represent the area of 135 square units a different way. Label the dimensions of the smaller regions in the area model.

2. Calculate the area of each of the two smaller regions. How does the sum of the two smaller regions compare to the total area of 135 square units?

3. Rewrite the original equation $5 \times 27 = 135$ with an equivalent equation to represent the model you drew.
   a. How can you rewrite the original product by substituting the sum of the two lengths making up the split side?
b. How can you rewrite the total area as the sum of the areas of the two smaller regions?

Think about other ways you could split one of the factors and write a corresponding equation. What would the equation look like if you split one of the factors into more than two regions?

4. Mark and label at least 2 more ways you could divide the area model. Write the corresponding equations. Then verify that the sum of the smaller regions is still equal to 135.

5. Reflect on the different ways you can rewrite the product of 5 and 27. Select one of your area models to complete the example.

\[ 5 \times 27 = 5(\_\_\_\_\_\_ + \_\_\_\_\_\_\_) \]

How did you split the side length of 27?

\[ = (5 \times \_\_\_\_\_) + (5 \times \_\_\_\_\_) \]

What are the factors of each smaller region?

\[ = \_\_\_\_\_\_ + \_\_\_\_\_\_ \]

What is the area of each smaller region?

\[ = \_\_\_\_ \]

What is the total area?
You just used the *Distributive Property*!

The **Distributive Property** of Multiplication over Addition states that for any numbers $a$, $b$, and $c$,

\[ a(b + c) = ab + ac. \]

6. Explain the Distributive Property using the area model shown.

**WORKED EXAMPLE**

An example of the Distributive Property.

\[ 4(2 + 15) = 4 \cdot 2 + 4 \cdot 15 \]

You can read and describe the expression $4(2 + 15)$ in different ways. For example, you can say:

- four times the quantity of two plus fifteen,
- four times the sum of two and fifteen, or
- the product of four and the sum of two and fifteen.

You can describe the expression $4(2 + 15)$ as a product of two factors. The quantity $(2 + 15)$ is both a single factor and a sum of two terms.

7. Write an equation in the form $a(b + c) = ab + ac$ for the other area models you created in this activity.
Tyler is setting up the gym floor for an after-school program. He wants to include a rectangular area for playing volleyball and another for dodgeball. He also wants to have an area for kids who like to play board games or just sit and read. The gym floor is 50 feet by 84 feet, or 4200 square feet.

1. Create a diagram to show how you would split up the gym floor. Represent your diagram using the Distributive Property and write an explanation for the areas assigned to each activity.

**TALK the TALK**

**Recognizing the Distributive Property**

Identify each statement as true or false. If the statement is false, show how you would rewrite it to make it a true statement.

1. **True**  **False**  \(3(2 + 4) = 3 \cdot 2 + 4\)

2. **True**  **False**  \(6(10 + 5) = 6 \cdot 10 + 6 \cdot 5\)

3. **True**  **False**  \(7(20 + 8) = 7 + 20 \cdot 8\)

4. **True**  **False**  \(4(5 + 10) = 20 + 10\)

5. **True**  **False**  \(2(6 + 11) = 12 + 22\)
Assignment

Write
Describe how the Distributive Property can be explained in terms of composing and decomposing numbers.

Remember
There are many ways to rewrite equivalent expressions using properties. The Distributive Property of Multiplication over Addition states that for any numbers $a$, $b$, and $c$, $a(b + c) = ab + ac$.

Practice
Decompose each rectangle into two or three smaller rectangles to demonstrate the Distributive Property. Then write each area in the form $a(b + c) = ab + ac$.

Evaluate each expression using the Distributive Property. Show your work.

4. $6(12 + 4)$
5. $10 + 4(2 + 20)$
6. $7(4 + 19)$
**Stretch**
Decompose each rectangle into two or three smaller rectangles to demonstrate the Distributive Property. Then write each area in the form $a(b + c) = ab + ac$.

1.  \[
\begin{array}{c}
\frac{1}{2} \\
6 \frac{1}{2}
\end{array}
\]

2.  \[
\begin{array}{c}
\frac{1}{3} \\
9 \frac{1}{6}
\end{array}
\]

**Review**
Calculate the area of each rectangle.

1. Width = 5 feet  
   Length = $\frac{2}{3}$ foot  
2. Width = 10 feet  
   Length = $\frac{2}{3}$ foot  
3. Width = 15 inches  
   Length = $\frac{2}{3}$ inch  
4. Width = 20 inches  
   Length = $\frac{2}{3}$ inch
ALL ABOUT THAT BASE . . . and Height

Area of Triangles and Quadrilaterals

WARM UP
Write 3 different expressions to describe the total area of this rectangle.

LEARNING GOALS
• State and compare the attributes of different shapes.
• Explain that the area of a parallelogram is the same as that of a rectangle with the same base length and height.
• Derive the formulas for the areas of triangles, parallelograms, and trapezoids by composing or decomposing the various shapes into rectangles, triangles, and other shapes.
• Apply the techniques of composing and decomposing shapes to solve real-world and mathematical problems.

KEY TERMS
• parallelogram
• altitude
• variable
• trapezoid

You can take a shape apart and put it back together in a different way without changing its area. How can you use composition and decomposition of shapes to reason about the areas of shapes and to derive formulas for the areas of common shapes?
In the 20s

Consider each figure.

1. Can you name each figure?

2. Describe the attributes of each shape. Are there any attributes that are shared across the different shapes?

3. Each shaded figure shown has an area of exactly 20 square units. Show how you know.
In this activity you will investigate the area of a parallelogram using what you know about the area of a rectangle. A parallelogram is a four-sided figure with two pairs of parallel sides and opposite sides that are equal in length.

1. Cut out a parallelogram from the grid at the end of the lesson.

2. Cut your parallelogram into pieces so that it can be reassembled to form a rectangle. Tape your rectangle in the space provided.

In a parallelogram, any of the four sides can be labeled as the base. The altitude of a parallelogram is another name for the height of a parallelogram. The altitude of a parallelogram is the perpendicular distance from the base of the parallelogram to the opposite side, represented by a line segment.

3. Label the base and height of the parallelogram and rectangle.
4. How does the height of the parallelogram relate to the height of the rectangle? How does the length of the base of the parallelogram relate to the length of the base of the rectangle? Explain your reasoning.

5. Describe the relationship between the areas of a parallelogram and rectangle that have the same base and height.

6. Use the terms base and height to describe how to calculate the area of a parallelogram.

In mathematics, one of the most powerful concepts is to use a letter to represent a quantity that varies, or changes. The use of letters, called variables, helps you write expressions to understand and represent problem situations.

7. Write the formulas to calculate the areas of a parallelogram and a rectangle. Use $b$ for base and $h$ for height.
The base of a triangle, like the base of a parallelogram, can be any of its sides. The height, or altitude, of a triangle is the length of a line segment drawn from a vertex of the triangle to the opposite side so that it forms a right angle with the opposite side.

Sailboat racecourses are often shaped like triangles. The course path is defined by buoys called marks. When the course is triangular, the marks are located at the corners, or vertices, of the triangle. Here is a sample course with the marks numbered.

Race officials need to know the area inside the course so that they can plan for the number of spectator boats that can anchor within.
1. Estimate the area of the triangular course in square units. Justify your estimate.

2. Use two sides of the triangle to draw a parallelogram on the grid. How does the area of the parallelogram relate to the area of the triangle?

3. Calculate the area enclosed by the triangular course.
4. Label a base and height of the original triangle in the diagram. Describe how to calculate the area of any triangle in terms of the base and the height.

5. Suppose you create a parallelogram using a different side of the triangle. Does this change the area of the triangle? Explain how you know.
You have seen that taking apart, or decomposing, a parallelogram forms a rectangle. And putting together, or composing, two triangles also forms a parallelogram. Composing and decomposing can help you think about the shapes differently in order to determine their areas. In this activity you will take apart and put together shapes to determine the formula for calculating the area of a trapezoid.

A trapezoid is a quadrilateral with two bases, often labeled $b_1$ and $b_2$. The bases are parallel to each other. The other two sides of a trapezoid are called the legs of the trapezoid. An altitude of a trapezoid is the length of a line segment drawn from one base to the other and perpendicular to both.

1. Label the bases of each trapezoid as $b_1$ and $b_2$.

Cut out two of the trapezoids at the end of the lesson to show how to determine each area.
2. Marcus cut out and composed two trapezoids into a parallelogram to figure out the exact area of one trapezoid. Show what Marcus did to determine the area.

3. Zoe folded the trapezoid so the bases aligned, cut along the fold, and rearranged the parts to form a parallelogram. Show what Zoe did to determine the area.

4. Angela decomposed the trapezoid into two triangles to determine the exact area. Use this trapezoid to recreate Angela’s strategy.

5. Describe how to calculate the area of any trapezoid in terms of the two bases and the height.
TALK the TALK

All Three Shapes

1. Draw each shape and then label a base and height. Next, write the formula to calculate the area of each. Use $A$ for the area, $b$ for the length of the base, and $h$ for the height.

- parallelogram
- triangle
- trapezoid

2. Show that the two triangles have the same area.

3. Write a paragraph that will convince your readers that the two triangles have the same area.
Shape Cut Outs

Extra shapes are included.
Assignment

Write
Define each term in your own words.

1. height of a parallelogram
2. height of a triangle

Remember
The area of a parallelogram, a triangle, or a trapezoid can be determined by composing or decomposing it into one or more shapes with an equal total area.

Area of a parallelogram = \(bh\)
Area of a triangle = \(\frac{1}{2}bh\)
Area of a trapezoid = \(\frac{1}{2}(b_1 + b_2)h\)

Practice
Answer each question for the given figure.

1. Identify a base and corresponding height for the given parallelogram. Determine the area of the parallelogram.

2. Calculate the area of the parallelogram.

3. Identify a base and corresponding height for the given triangle. Determine the area of the triangle.

4. Calculate the area of the triangle.

LESSON 2: All About That Base . . . and Height   •   M1-27
5. Identify the two bases and the height for the given trapezoid. Determine the area of the trapezoid.

6. Yvonne cut a picture into the shape of a trapezoid to place into her scrapbook. The picture is shown. What is the area of the picture?

**Stretch**

1. What is the area of a parallelogram that has a base of $\frac{3}{4}\text{ ft}$ and a height of $1\frac{1}{3}\text{ ft}$?
2. Calculate the area of the triangle.

**Review**

Use the Distributive Property to write an equivalent addition expression for each.

1. $6(9 + 1)$
2. $(14 + 3)7$
3. $\frac{1}{2}(7 + 10)$

Decompose each rectangle into two or three smaller rectangles to demonstrate the Distributive Property. Then write each in the form $ab + ac$. 

4. 

5. 

M1-28 • **TOPIC 1**: Factors and Area
Slicing and Dicing
Composite Figures

WARM UP
Use a formula to determine the area of each figure.

1.  
   \[ \text{Area} = \frac{1}{2} \times \text{base} \times \text{height} \]
   \[ = \frac{1}{2} \times 10.5 \text{ m} \times 22 \text{ m} \]

2.  
   \[ \text{Area} = \frac{1}{2} \times \text{base} \times \text{height} \]
   \[ = \frac{1}{2} \times 7 \text{ cm} \times 11 \text{ cm} \]

3.  
   \[ \text{Area} = \frac{1}{2} \times \text{base} \times \text{height} \]
   \[ = \frac{1}{2} \times 30 \text{ yd} \times 65 \text{ yd} \]

LEARNING GOALS
- Decompose composite geometric figures into rectangles, parallelograms, and/or triangles to determine their areas.
- Solve real-world problems by decomposing shapes into triangles and rectangles.

KEY TERM
- kite

You know how to calculate the area of triangles, rectangles, parallelograms, and trapezoids. How can you use what you know about the areas of these shapes to determine areas of more complex shapes?
Let Me Count the Ways

To calculate the total area of an oddly-shaped region, one strategy is to divide the region into smaller familiar regions and then calculate the area of each familiar region.

1. Draw lines in each figure to divide the figure into smaller familiar shapes. Then name the familiar shapes that make up the total figure.

   a.

   ![Image of figure a]

   b.

   ![Image of figure b]
A kite is a quadrilateral with two pairs of consecutive congruent sides where opposite sides are not congruent.

The area of a kite, like that of other quadrilaterals, can be determined by decomposing its shape into smaller familiar shapes.

Mr. Gram sketched the kite shown. He asked his students to add a line segment that would divide the kite into two familiar figures.

1. Describe each student’s strategy and identify any additional information you would need to calculate the area. Would you rather use Molly’s or James’s diagram to compute the area of the kite?
2. Use the information given to calculate the area of the kite using both Molly’s and James’s strategies.

Given:
- \( AC = 5 \text{ cm} \)
- \( AE = 1.1 \text{ cm} \)
- \( CE = 3.9 \text{ cm} \)
- \( BD = 2.5 \text{ cm} \)
- \( BE = 1.25 \text{ cm} \)
- \( DE = 1.25 \text{ cm} \)

3. Which method do you prefer for Question 2, Molly’s or James’s method? Why?
Decompose the composite shape in each image into parallelograms, triangles, and/or trapezoids to calculate the approximate area of each. Show your work.

1. Suppose a gallon of paint covers about 400 square feet. How much paint would you need to paint the entire back of this house?

2. Suppose that carpeting costs $1.20 per square foot. How much would it cost to carpet every room in this house except the kitchen?
3. Estimate the area of France.

4. Estimate the area of Namibia.
5. The figure shown is composed of a rectangle and four congruent trapezoids. Determine the area of the shaded region.

![Diagram of a rectangle and four congruent trapezoids]

6. The figure shown is composed of a rectangle and a hexagon. The length of each side of the hexagon is 2 centimeters. Determine the area of the shaded region.

![Diagram of a rectangle and a hexagon]
TALK the TALK

Use Your Powers of Mathematical Reasoning

1. Determine the area of the shaded triangle inside the square. Explain your strategy.

2. Create a presentation of your solution strategy for the class.
Write
Define composite figure and draw a picture of an example.

Remember
The area of a composite figure can be determined by decomposing it into familiar shapes and then adding together the areas of those shapes.

Practice
1. Calculate the area of the composite figure.

2. A city wants to create a garden according to the plan below. Calculate the area of the garden.

3. In the given kite, $AE = 6$ ft, $CE = 6$ ft, $BE = 9$ ft, and $DE = 15$ ft. Determine the area of the kite.

4. In the given kite, $SZ = 10$ yards, $WZ = 10$ yards, $TZ = 12$ yards, and $RZ = 32$ yards. Determine the area of the kite.
Stretch
Calculate the area of each shaded region.

1. The figure is composed of 2 kites.
   Given: $MR = RS = 5$ feet, $ST = PT = 10$ feet, and $NS = QS = 12$ feet.

2. The figure is composed of 2 congruent triangles and a rhombus.

Review
1. A patio is built in the shape of a trapezoid, as shown. Determine the area of the patio.

2. Calculate the area of the given triangle.

3. Calculate the area of the given parallelogram.

4. Determine the area of a square picture that has a side length of 14 cm.

5. Use the Distributive Property to write an equivalent addition expression for $5(17 + 20)$. 

M1-38 • TOPIC 1: Factors and Area
WARM UP
List all factor pairs for each number.

1. 42
2. 56
3. 84
4. 91

LEARNING GOALS
• Identify the factors of numbers and the common factors of two whole numbers.
• Identify the multiples of numbers and the common multiples of two whole numbers.
• Write and evaluate numeric expressions using the Distributive Property to model composing and decomposing the areas of rectangles.
• Rewrite the sum of two whole numbers with a common factor as a product using the Distributive Property.

KEY TERMS
• common factor
• relatively prime
• greatest common factor (GCF)
• multiple
• Commutative Property of Multiplication
• least common multiple (LCM)

Just as you can compose and decompose shapes, you can compose and decompose numbers using factors and multiples. How can you use shapes to see relationships between numbers?
Getting Started

How Many Rectangles Can You Build?

Understanding the area of rectangles is helpful when learning about factors. A rectangular area model is one way to represent multiplication.

Your class is going to create area models for each number: 12, 15, 16, and 20. For the number assigned to you by your teacher, use the grid paper at the end of the lesson to create and cut out as many unique rectangles as possible with the area of your assigned number. Label each rectangle with its dimensions.

Number assigned to me: __________

1. List the dimensions of all of the rectangles that you created for your assigned number.

2. How do you know if you have created all of the possible rectangles with the given area?

3. How are factors represented in your rectangles?

4. List all of the factors of the number that you were assigned.
ACTIVITY 4.1

Using Rectangles to Determine Common Factors

For this investigation, select a partner who has created area models for a number different from the number assigned to you.

Together with your partner, combine one of your rectangles and one of your partner’s rectangles to make a bigger rectangle. If possible, use this method to create additional rectangles.

1. Complete the table with the information from each larger rectangle created by you and your partner.

<table>
<thead>
<tr>
<th>Number assigned to me</th>
<th>Number assigned to my partner</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensions of Smaller Rectangle 1</td>
<td>Dimensions of Smaller Rectangle 2</td>
</tr>
<tr>
<td>$l \times w_1$</td>
<td>$l \times w_2$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. How are the dimensions of the larger rectangle related to its total area?

3. For each larger rectangle you and your partner created, write a numeric expression that relates the dimensions of the larger rectangle to the sum of the areas of the smaller rectangles.
Consider any factors that are shared between your number and your partner’s number. These are called **common factors**.

4. **How are the common factors represented in the larger rectangles that you and your partner created?**

5. **How are the common factors represented in the numeric expressions that you and your partner wrote?**

6. **List the common factors of the two numbers.**

---

**WORKED EXAMPLE**

One way to determine common factors is to use prime factorization. Start by writing each number as a product of its prime factors.

$56 = 2 \cdot 2 \cdot 2 \cdot 7$

$42 = 2 \cdot 3 \cdot 7$

Organize the prime factors into a table, where only shared factors are listed in the same column.

<table>
<thead>
<tr>
<th>Number</th>
<th>Prime Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>2 2 2 7</td>
</tr>
<tr>
<td>42</td>
<td>2 3 7</td>
</tr>
</tbody>
</table>

The common factors of the two numbers are the numbers that are in both rows and the product of the numbers that are in both rows.

The common factors of 56 and 42 are 2, 7, and 14.
1. How do you know that 14 is a common factor of 56 and 42?

2. Why is there a space between 2 and 7 in the top row of the table?

3. Create a table to identify common factors.
   a. Identify all of the common factors of 54 and 84.

   b. Of the common factors, which factor is the largest?

   The **greatest common factor (GCF)** is the largest factor two or more numbers have in common.

4. Rewrite each numeric expression using the Distributive Property and the GCF.
   a. 56 + 42

   b. 54 + 84

Two numbers that do not have any common factors other than 1 are called **relatively prime.**
Rectangular arrays can also be used to determine multiples and common multiples.

Consider the area model for $6 \cdot 8 = 48$.

One way to think about the area model is to analyze the collection of columns. As you look at how the area model builds from left to right, the addition of each new column creates a multiple of 6. So, column 1 alone is a $6 \times 1$ rectangle, which represents the first multiple of 6, which is 6. By adding column 2, the rectangle is now $6 \times 2$, which represents the second multiple of 6, which is 12. The whole rectangle represents $6 \times 8$, or 48.

1. List the first eight multiples of 6 by labeling each column of the area model.

Next, think about the area model as a collection of 6 rows. The first row alone creates an $8 \times 1$ rectangle, which represents the first multiple of 8, which is 8. Including all rows of the $8 \times 6$ rectangle represents the sixth multiple of 8, which is 48.

2. List the first six multiples of 8 by labeling each row of the area model.
While 48 is a multiple shared by both 6 and 8, it is not the **least common multiple (LCM)**. The LCM is the smallest multiple (other than zero) that two or more numbers have in common.

3. **Analyze the multiples of 6 and 8 that you labeled on the area model. Identify the least common multiple of 6 and 8.**

As demonstrated by the rectangular array, for any two whole numbers \(a\) and \(b\), a common multiple is \(a \cdot b\). However, this number may not be the **least** common multiple of \(a\) and \(b\).

4. **Determine the least common multiple of 6 and 9.**
   a. List the first 9 multiples of 6.
   b. List the first 6 multiples of 9.
   c. What is the least common multiple of 6 and 9?

5. **Determine the least common multiple of 7 and 8.**

6. **Using prime factorization, how can you determine whether the least common multiple of two numbers is the product of the two numbers, or is less than the product of the two numbers?**
TALK the TALK

Bringing It Back Around

Answer each question to show how to use the Distributive Property to decompose numbers.

1. Consider the sum $36 + 24$.
   
   a. Express the sum $36 + 24$ as many ways as possible as the product $a(b + c)$.

   b. How can you use factors to determine if you have listed all possible products $a(b + c)$ that are equivalent to $36 + 24$?

2. Suppose you have a composite figure composed of a rectangle and another parallelogram with a shared side. The area of the rectangle is 72 square centimeters and the area of the parallelogram is 84 square centimeters.

   Explain how to use factors and multiples to determine all possible dimensions $a$, $b$, and $h$ for the figure.
Grid Paper
Assignment

Write

1. Match each definition to its corresponding term.
   a. a rectangular arrangement that has an equal number of objects in each row and an equal number of objects in each column
   b. the product of a given whole number and another whole number
   c. two natural numbers other than zero that are multiplied together to produce another number
   d. one of the two numbers being multiplied together in a factor pair
   e. changing the order of two or more factors in a multiplication problem does not change the product

   i. factor pair
   ii. array
   iii. Commutative Property of Multiplication
   iv. factor
   v. multiple

2. Select the word that makes the following statement true. Then, use complete sentences to explain your choice: The LCM of two numbers is (always, sometimes, never) the product of the two numbers.

Remember

Numbers can be decomposed into a product of their prime factors. Numbers can be composed into multiples. Numbers can be compared by their greatest common factor and their least common multiple.

Practice

1. Consider the numbers 18 and 30.
   a. List all of the factors of 18.
   b. List all of the factors of 30.
   c. What factors do 18 and 30 have in common?
   d. What is the greatest common factor of 18 and 30?

2. Consider the numbers 54 and 72.
   a. Complete a prime factorization of 54 and write it as a product of primes.
   b. Complete a prime factorization of 72 and write it as a product of primes.
   c. Put the prime factors of 54 and 72 into a table.
   d. What are the common factors of 54 and 72?
   e. What is the greatest common factor of 54 and 72?

3. For each pair of numbers, determine the least common multiple and at least one other common multiple.
   a. 3 and 5
   b. 4 and 6
   c. 8 and 12
**Stretch**

1. Determine the LCM for each group of numbers.
   a. 4, 8, 14
   b. 9, 15, 18

2. Determine the GCF for each group of numbers.
   a. 8, 27, 35
   b. 20, 90, 50

**Review**

Determine the area of each figure.

1. 

   ![Rectangle](image)

   12 m

   4 m

   6 m

   16 m

2. In the given kite, $SZ = WZ = 10$ yards, $TZ = 12$ yards, and $RZ = 32$ yards.

   ![Kite](image)

3. The polygon is a rhombus.

   ![Rhombus](image)

4. 

   ![Triangle](image)

   2 yards

   5 yards

   8 yards

5. 

   ![Triangle](image)

   7 ft

   7 ft
LEARNING GOALS
• Determine the greatest common factor of two whole numbers less than or equal to 100.
• Use greatest common factors and the Distributive Property to rewrite the sum of whole numbers 1–100.
• Determine the least common multiple of two whole numbers less than or equal to 12.

Number relationships are useful in solving everyday problems and in mental arithmetic. Understanding these relationships will deepen your knowledge of how the number system is structured. How can you use LCM and GCF to compose and decompose numbers?
Beads, Beads, Beads

Emily has three bags of different types of beads. She wants to split up the beads into mixed packages to share with her friends. She wants each package to have exactly the same number of each type of bead with no beads left over.

1. What is the greatest number of packages that Emily can assemble? Describe the collection of beads in each package.

Spacers 40 count
Round Beads 72 count
Rectangular Beads 24 count
In the previous activity, you determined the greatest number of packages that Emily could make from three different types of beads. In other words, you were looking for the greatest number that is a factor of the three other numbers (40, 72, and 24).

One way to determine the greatest common factor is to start by listing all the prime factors of each number. The table shows the prime factorization of 24, 40, and 72.

<table>
<thead>
<tr>
<th>Number</th>
<th>Prime Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>2 2 2 3</td>
</tr>
<tr>
<td>40</td>
<td>2 2 2 5</td>
</tr>
<tr>
<td>72</td>
<td>2 2 2 3 3</td>
</tr>
</tbody>
</table>

1. Analyze the table and circle the common prime factors.

The greatest common factor is the product of the common prime factors.

2. What is the greatest common factor of 24, 40, and 72?

3. What is the least common factor of 24, 40, and 72?

4. Rewrite the numeric expression $24 + 40 + 72$ using the Distributive Property and the GCF.
5. Determine the greatest common factor of each pair.
   a. 36 and 48  
   b. 37 and 81

6. Rewrite each numeric expression using the Distributive Property and the GCF.
   a. 36 + 48  
   b. 37 + 81

Remember that common factors help you think about how to divide, or share things equally. Common multiples help you think about how things with different cycles can occur at the same time.

1. Ramon and Justine are watching different broadcasts of a parade on television. The broadcast Ramon is watching airs commercials every 17 minutes. The broadcast Justine is watching airs commercials every 14 minutes. Both broadcasts started at 7:00 a.m. and are scheduled to end at 9:00 a.m. When will commercials air on both broadcasts at the same time? Explain your reasoning.
2. Two cyclists ride on the same circular path. The first rider completes a lap in 12 minutes. The second rider completes a lap in 18 minutes. Both riders start at the starting line at the same time and go in the same direction. If the riders maintain their speed, after how many minutes will they meet again at the starting line? Explain your reasoning.

3. Dr. Abramson and her assistants are working on three different experiments using water. Each experiment lasts for 15 minutes. For the first experiment, the water level must be checked every 12 seconds. For the second experiment, the temperature of the water must be checked every 30 seconds. For the third experiment, the color of the water must be checked every 36 seconds. In minutes, list the times all three experiments will need to be checked at the same time.

4. The students in Mr. Michael's art class are decorating a booth for Harvest Day. They have blue cloth that is 60 inches long, gold cloth that is 48 inches long, and white cloth that is 72 inches long. They want to cut all of the cloth into pieces of equal length.

   a. What is the greatest possible length of the pieces without having any cloth left over? Explain your reasoning.

   b. How many pieces of each color cloth will they have?
5. Boxes that are 16 inches tall are being stacked next to boxes that are 20 inches tall.

a. What is the shortest height at which the two stacks will be the same height? Explain your reasoning.

b. How many boxes will be in each stack?

Making Connections

ACTIVITY 5.3

Making Connections

Recall that numbers that are relatively prime have no common factors other than 1. These number pairs can show interesting patterns.

1. For each pair of numbers, determine their product, their least common multiple, and their greatest common factor.

   a. 12 and 10
   b. 9 and 15

   c. 9 and 10
   d. 5 and 9
2. Consider the GCF and LCM of the pair 9 and 10 and the pair 5 and 9.
   
   a. What relationship do you notice between the product, LCM, and GCF of the pairs of numbers?

   b. Write a sentence to describe your conjecture. Test your conjecture by determining the product, LCM, and GCF of additional pairs of numbers between 1 and 20.
TALK the TALK

In Summary

Answer each question to summarize what you know about greatest common factors and least common multiples.

1. Can you always determine the greatest common factor of any two numbers? Explain your reasoning.

2. If the greatest common factor of two numbers is 1, what can you say about the numbers?

3. Can you always determine the least common multiple of any two numbers? Explain your reasoning.

4. If the least common multiple of two numbers is the product of those numbers, what can you say about the two numbers?

5. How can you use the GCF and the Distributive Property to rewrite the sum of two numbers?
Assignment

Write
Write a definition for each term in your own words.

1. least common multiple (LCM)
2. greatest common factor (GCF)

Remember
Common factors help determine how to divide or share things equally. Common multiples help determine how things with different cycles can occur at the same time.

Practice
1. Ronna is a quality control engineer in a car parts factory. Part of her job is to make sure the parts are the right size.
   a. In one section of the factory, two machines mold different parts that will eventually be put together in an assembly plant. The first machine makes a part every 12 seconds, and the second machine makes a part every 45 seconds. Ronna decides to test these parts each time they both come out of the machines at the same time. How often does Ronna test the parts? Show your work and express your answer in minutes.

2. Mr. Ellis runs an after-school program for nine- and ten-year-olds. Each day the children participate in an activity or sport and receive a snack. One afternoon, 56 nine-year-olds and 42 ten-year-olds attend the after-school program.
   a. Mr. Ellis wants to divide the group into basketball teams so that each team has the same number of nine-year-olds, and each team has the same number of ten-year-olds. How many different ways can he divide the group?
   b. What is the greatest number of teams Mr. Ellis can make so each team has the same number of 9-year-olds and the same number of 10-year-olds?
   c. Do you think Mr. Ellis should make the greatest number of teams he can? Explain your reasoning.

Stretch
1. Create your own word problem that requires an LCM to solve. Show the solution.
2. Create your own word problem that requires a GCF to solve. Show the solution.
Review

1. Consider the numbers 18 and 28.
   a. Sketch all distinct area models for the number 18.
   b. Sketch all distinct area models for the number 28.
   c. Use your area models and the Distributive Property to rewrite the expression 18 + 28.

2. The base of the triangle is labeled. Draw a segment that represents the height of the triangle.

3. Determine the area of the given figure composed of a parallelogram and a triangle.
Factors and Area Summary

KEY TERMS
- numeric expression
- equation
- Distributive Property
- parallelogram
- altitude
- variable
- trapezoid
- kite
- common factor
- relatively prime
- greatest common factor (GCF)
- multiple
- Commutative Property of Multiplication
- least common multiple (LCM)

LESSON 1
Taking Apart Numbers and Shapes

A numeric expression is a mathematical phrase that contains numbers and operations. An equation is a mathematical sentence that uses an equals sign to show that two quantities are the same as one another.

There are many ways to rewrite equivalent expressions using properties. The Distributive Property states that for any numbers $a$, $b$, and $c$, $a(b + c) = ab + ac$.

For example, you can use the Distributive Property to rewrite the expression $4(2 + 15)$.

$$4(2 + 15) = 4 \cdot 2 + 4 \cdot 15$$
A **parallelogram** is a four-sided figure with two pairs of parallel sides, with each pair equal in length.

The **altitude** is the height of a geometric figure. In a parallelogram, it is the perpendicular distance from the base to the opposite side. The area of a parallelogram is equal to $b \cdot h$, where the variable $b$ represents the base and $h$ represents the height. A **variable** is a letter that is used to represent a number.

For example, in this parallelogram, the base, $b$, is 20 feet and the altitude, or height, $h$, is 12 feet.

$$\text{Area of a parallelogram} = bh$$
$$= (20)(12)$$
$$= 240 \text{ square feet}$$

The area of a triangle is equal to $\frac{1}{2} bh$. The base of a triangle can be any of its sides.

The height, or altitude, of a triangle is the length of a line segment drawn from a vertex of the triangle to the opposite side so that it forms a right angle with the opposite side.

For example, in this triangle, the base, $b$, is equal to 2 yards and the altitude, or height, $h$, is equal to 1.5 yards.

$$\text{Area of a triangle} = \frac{1}{2}bh$$
$$= \frac{1}{2}(2)(1.5)$$
$$= 1.5 \text{ square yards}$$
A **trapezoid** is a quadrilateral with two bases, often labeled \(b_1\) and \(b_2\). The bases are parallel to each other. The height is the perpendicular distance between the bases. The area of a trapezoid is equal to \(\frac{1}{2}(b_1 + b_2)h\).

For example, in this trapezoid, one of the bases is 8 meters and the other base is 12 meters. The altitude, or height, \(h\), of the trapezoid is 9 meters.

\[
\text{Area of trapezoid} = \frac{1}{2}(b_1 + b_2)h \\
= \frac{1}{2}(8 + 12)(9) \\
= \frac{1}{2}(20)(9) \\
= 90 \text{ square meters}
\]

A **kite** is a quadrilateral with two pairs of consecutive congruent sides where opposite sides are not congruent. The area of a kite, like that of other quadrilaterals, can be determined by decomposing its shape into smaller familiar shapes.

Area is additive. The area of a composite figure can be determined by decomposing it into familiar shapes and then adding together the areas of those shapes.

For example, in Kite \(ABCD\), \(BE = ED = 1.25 \text{ cm}\), and \(AC = 5 \text{ cm}\).

\[
\text{Area of Kite } ABCD = \text{Area of Triangle } ABC + \text{Area of Triangle } ADC \\
= \frac{1}{2}(5)(1.25) + \frac{1}{2}(5)(1.25) \\
= 3.125 + 3.125 \\
= 6.25
\]

The area of Kite \(ABCD\) is 6.25 square centimeters.
When two or more numbers are factored, any factors that the numbers share are **common factors**. Two numbers that do not have any common factors other than 1 are called **relatively prime**. The **greatest common factor (GCF)** is the largest factor two or more numbers have in common.

One way to determine common factors is using prime factorization.

For example, you can use prime factorization to determine common factors of 56 and 42. Start by writing each number as a product of its prime factors.

Organize the prime factors into a table, where only shared factors are listed in the same column.

<table>
<thead>
<tr>
<th>Number</th>
<th>Prime Factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>2 2 7</td>
</tr>
<tr>
<td>42</td>
<td>2 3 7</td>
</tr>
</tbody>
</table>

The common factors of the two numbers are the numbers that are in both rows and the product of the numbers that are in both rows.

The common factors of 56 and 42 are 2, 7, and 14.

The greatest common factor of 56 and 42 is 14.

A **multiple** is the product of a given whole number and another whole number. The **Commutative Property of Multiplication** states that for any numbers a and b, the product a \* b is equal to the product b \* a. The first eight multiples of 6 and 8 are given.

- 6: 6, 12, 18, 24, 30, 36, 42, 48
- 8: 8, 16, 24, 32, 40, 48, 56, 64

The **least common multiple (LCM)** is the smallest multiple (other than zero) that two or more numbers have in common. The LCM of 6 and 8 is 24.
The greatest common factor is the product of the common prime factors.

For example, the GCF of 24, 40, and 72 is $2 \times 2 \times 2 = 8$.

When using GCF and LCM to solve problems, remember that common factors help you think about how to divide, or share things equally, and common multiples help you think about how things with different cycles can occur at the same time.

For example, a local bus arrives at the stop near Aaron’s house every 15 minutes. An express bus arrives at the same stop every 9 minutes. Aaron sees both a local and an express bus arrive at the stop at 10 A.M. What is the next time that he would expect to see both buses arrive at the stop?

The problem is asking about when the two different cycles of the buses will occur again at the same time, so you can use the least common multiple of 15 and 9 to answer the question.

The multiples of 15 are 15, 30, 45, 60, 75, . . .
The multiples of 9 are 9, 18, 27, 36, 45, 54, . . .

The least common multiple of 15 and 9 is 45, therefore the two buses should arrive at the stop at the same time every 45 minutes. The next time Aaron would expect to see both buses at the stop is 10:45 A.M.
TOPIC 2

Positive Rational Numbers

A flood gauge measures how far above normal the surface of the water is.

Lesson 1
Thinking Rationally
Identifying and Ordering Rational Numbers ................................. M1-71

Lesson 2
Did You Get the Part?
Multiplying and Dividing with Fractions ................................. M1-83

Lesson 3
Yours IS to Reason Why!
Fraction by Fraction Division .................................................. M1-93
TOPIC 2: POSITIVE RATIONAL NUMBERS
The focus of this topic is fraction division. Students review fraction and decimal comparisons and multiplying with fractions prior to working with fraction division. Algorithms for fraction division are addressed in this topic, but bear in mind that students may not achieve fluency within the timeline allowed for this topic. Fluency requires time and practice. Although this topic represents the culmination of students’ learning about operations with fractions, they will continue to develop fluency with fraction operations throughout the course.

Where have we been?
Students began their formal study of fractions in grade 3. They understand fractions as numbers and can reason about relative sizes of fractions. They have learned to add, subtract, and multiply fractions. Students also know how to divide whole numbers by unit fractions (e.g., $6 \div \frac{1}{2}$) and unit fractions by whole numbers (e.g., $\frac{1}{4} \div 3$).

Where are we going?
By learning multiple division strategies and using estimation and mental strategies, students can choose the most efficient strategy for a given problem.

Throughout grade 6, students will operate with positive rational numbers. Students who have mastered plotting and ordering positive rational numbers on a number line will be prepared to plot and order the full set of rational numbers on a number line and as pairs on a coordinate plane.

Using Bar Models to Represent Quotients with Fractions
A bar model can show the quotient of two fractions, such as $\frac{3}{4} \div \frac{1}{4}$. The division expressions asks, how many $\frac{1}{4}$s are in $\frac{3}{4}$?

There are 3 one-fourths in $\frac{3}{4}$, so $\frac{3}{4} \div \frac{1}{4} = 3$. 
Myth: "If I can get the right answer, then I should not have to explain why."

Sometimes you get the right answer for the wrong reasons. Suppose a student is asked “What is 4 divided by 2?” and she confidently answers “2!” If she does not explain any further, then it might be assumed that she understands how to divide whole numbers. But, what if she used the following rule to solve that problem? “Subtract 2 from 4 one time.” Even though she gave the right answer, she has an incomplete understanding of division.

However, if she is asked to explain her reasoning, by drawing a picture, creating a model, or giving a different example, the teacher has a chance to remediate her flawed understanding. If teachers aren’t exposed to their students’ reasoning for both right and wrong answers, then they won’t know about or be able to address misconceptions. This is important because mathematics is cumulative: new lessons build upon previous understandings.

Ask your student to explain his or her thinking, when possible, even if you don’t know whether the explanation is correct. When children (and adults) explain something to someone else, it helps them learn. Just the process of trying to explain is helpful.

#mathmythbusted

Talking Points
You can support your student’s learning by practicing with them. Students are learning to divide with fractions. This involves dividing whole numbers by fractions, fractions by fractions, and fractions by whole numbers.

Some Things to Look For
When comparing two fractions, students often assume that the fraction with the smaller denominator must be the smaller fraction. This is not always true!

Remind your student to take their time as they work with fractions. Fractions can be tricky!

Key Terms

benchmark fraction
Benchmark fractions are common fractions, like \( \frac{1}{2} \) or \( \frac{1}{4} \), you can use to estimate the value of other fractions.

complex fraction
A fraction is complex if it has a fraction in numerator, denominator, or both.

multiplicative inverse
The multiplicative inverse of a number \( \frac{a}{b} \) is the number \( \frac{b}{a} \), where \( a \) and \( b \) are nonzero numbers.
LEARNING GOALS
- Understand that counting numbers, fractions, and decimals are rational numbers.
- Identify properties of rational numbers.
- Identify models for rational numbers.
- Fluently compare and order rational numbers.

KEY TERMS
- positive rational number
- benchmark fraction

You have learned about whole numbers, fractions, and decimals. How can you compare these types of numbers?
Getting Started

How Many Can You Name?

You have learned about many different types of numbers. List as many types of numbers as you can. Give an example of each number type.

ACTIVITY 1.1 Identifying Positive Rational Numbers

You can group numbers in many different ways.

1. Cut out the cards at the end of this lesson. Sort the cards into different groups. You may sort the cards in any way you think is appropriate, but you must sort them into more than 1 group. Give each group of cards a title.

   Explain how you sorted the numbers and diagrams on the cards, including why you gave each group its title.

2. Compare your groupings with your classmates’ groupings. Create a list of some of the different ways to group the numbers.
3. Vivianne grouped these cards together. What reason could she give for why she put these cards into the same group?

\[
\begin{array}{c}
\frac{1}{5}, 0.2 \\
\end{array}
\]

4. Danika and Josh explained how they sorted the numbers.

Danika
I grouped these numbers together because they all represent whole numbers.

\[
\begin{array}{c}
8, 5, 10, 0, 1, 0 \\
\end{array}
\]

a. Show why Danika’s reasoning is correct.

b. Identify other numbers or diagrams that belong in Danika’s group.

Josh
I grouped these numbers together because they are all equal.

\[
\begin{array}{c}
3, 3, 3 \\
\frac{5}{5}, 4, 5 \\
\end{array}
\]

c. Explain why Josh’s reasoning is not correct.

d. Identify pairs of cards which show equal values. How many pairs can you find?
A **positive rational number** is a number that can be written in the form \( \frac{a}{b} \), where \( a \) and \( b \) are both whole numbers greater than 0.

**WORKED EXAMPLE**

Is 0.75 a rational number?

To write a decimal like 0.75 in the form \( \frac{a}{b} \), where \( a \) and \( b \) are both whole numbers and \( b \) is not equal to 0:

- **Read the decimal using place value.**
  
  \[
  0.75 \quad \text{seventy-five hundredths}
  \]

- **Write the decimal as a fraction.**
  
  \[
  0.75 \quad \frac{75}{100}
  \]

The fraction \( \frac{75}{100} \) is written in the form \( \frac{a}{b} \), where \( a \) is equal to 75 and \( b \) is equal to 100. The numbers 75 and 100 are both whole numbers greater than 0.

So, 0.75 is a rational number.

---

1. **Show that the decimals 0.6, 0.1, 0.2, and 0.325 are positive rational numbers.**

2. **Which numbers, if any, that you sorted are not positive rational numbers? Explain your answer.**
Benchmark fractions are common fractions you can use to estimate the value of fractions.

Three common benchmark fractions are $\frac{0}{1}$, $\frac{1}{2}$, and $\frac{1}{1}$.

A fraction is close to 0 when the numerator is very small compared to the denominator.

A fraction is close to $\frac{1}{2}$ when the numerator is about half the size of the denominator.

A fraction is close to 1 when the numerator is very close in size to the denominator.

1. Name the closest benchmark fraction for each fraction given.
   a. $\frac{4}{9}$
   b. $\frac{8}{9}$
   c. $\frac{6}{9}$
   d. $\frac{5}{67}$
   e. $\frac{7}{15}$
   f. $\frac{7}{12}$
   g. $\frac{5}{6}$
   h. $\frac{14}{27}$
   i. $\frac{12}{13}$
   j. $\frac{1}{17}$
   k. $\frac{5}{11}$
   l. $\frac{3}{7}$

2. Write the unknown numerator or denominator so that each fraction is close to but greater than 0.
   a. $(\_\_\_\_) \quad \frac{12}{\_\_\_\_}$
   b. $(\_\_\_\_) \quad \frac{27}{\_\_\_\_}$
   c. $\frac{8}{\_\_\_\_}$
   d. $\frac{7}{\_\_\_\_}$
3. Write the unknown numerator or denominator so that each fraction is close to but less than $\frac{1}{2}$.
   a. $\left( \frac{\_}{12} \right)$  
   b. $\left( \frac{\_}{27} \right)$  
   c. $\left( \frac{8}{\_} \right)$  
   d. $\left( \frac{7}{\_} \right)$

4. Write the unknown numerator or denominator so that each fraction is close to but less than 1.
   a. $\left( \frac{\_}{12} \right)$  
   b. $\left( \frac{\_}{27} \right)$  
   c. $\left( \frac{8}{\_} \right)$  
   d. $\left( \frac{7}{\_} \right)$

5. Describe the relationship between $a$ and $b$ when the fraction $\frac{a}{b}$ is:
   a. close to 0.  
   b. close to $\frac{1}{2}$.  
   c. close to 1.

6. Compare each pair of fractions using benchmark fractions. Insert a $>$ or $<$ symbol to make the inequality true. Explain your reasoning.
   a. $\frac{11}{12}$ ____ $\frac{5}{9}$  
   b. $\frac{5}{9}$ ____ $\frac{5}{7}$  
   c. $\frac{7}{13}$ ____ $\frac{5}{11}$  
   d. $\frac{5}{10}$ ____ $\frac{7}{10}$

7. Compare the fractions in each pair. Think about how close the fractions are to 0, $\frac{1}{2}$, or 1.
   a. $\frac{5}{8}$ and $\frac{7}{12}$  
   b. $\frac{14}{15}$ and $\frac{7}{8}$  
   c. $\frac{1}{9}$ and $\frac{1}{23}$
Felipe and Corinne ordered the rational numbers 0.8, 0.06, and $\frac{3}{5}$ from least to greatest using different strategies. Felipe used benchmark numbers, and Corinne used equivalent fractions.

1. Use Felipe’s strategy of benchmark numbers to order the rational numbers from least to greatest.

2. Use Corinne’s strategy of equivalent fractions to order the rational numbers from least to greatest.

3. Use any strategy to order the rational numbers 0.6, $\frac{3}{4}$, and $\frac{5}{8}$ from least to greatest.

4. List the fractions in each set in ascending order.
   a. $\frac{1}{8}, \frac{1}{11}, \frac{1}{9}, \frac{1}{4}, \frac{1}{7}, \frac{1}{5}$
   b. $\frac{4}{5}, \frac{4}{10}, \frac{4}{12}, \frac{4}{7}$
   c. $\frac{3}{8}, \frac{3}{11}, \frac{3}{9}, \frac{3}{4}, \frac{3}{7}, \frac{3}{5}$

5. What do the fractions in each part of Question 4 have in common? Explain how you determined the order of the fractions in each.
TALK the TALK

Close to Half

Consider the fractions shown.

\[
\frac{5}{9}, \frac{7}{13}, \frac{2}{7}, \frac{10}{11}
\]

1. Write the fractions in ascending order. Use what you know about benchmark fractions to determine the order. Explain your reasoning.
<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1/12</td>
<td>3/5</td>
<td>5/6</td>
<td>3/4</td>
<td>1/5</td>
</tr>
<tr>
<td>1/3</td>
<td>3/8</td>
<td>1/4</td>
<td>2/3</td>
<td>4/5</td>
</tr>
<tr>
<td>5/8</td>
<td>8/8</td>
<td>8/4</td>
<td>5/5</td>
<td>10/5</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.1</td>
<td>0.2</td>
<td>0.06</td>
</tr>
<tr>
<td>0.75</td>
<td>0.5</td>
<td>1</td>
<td>0</td>
<td>0 1</td>
</tr>
</tbody>
</table>
Assignment

Write
Describe a way to compare two positive rational numbers that are not written in the same form.

Remember
A positive rational number is a number that can be written in the form $\frac{a}{b}$, where $a$ and $b$ are both whole numbers, and $b$ is not equal to 0.

An inequality is a statement that one number is less than or greater than another number.

Practice
Order the rational numbers in each group from least to greatest.
1. 0.09, 0.1, $\frac{2}{25}$
2. $\frac{5}{6}$, $\frac{3}{8}$, $\frac{2}{2}$
3. 0.55, $\frac{2}{5}$, $\frac{2}{3}$
4. 4.2, 3.10, $\frac{4}{5}$, 3.01, 2.3, $\frac{2}{5}$, 3.017
5. 6.84, $\frac{8}{5}$, 6.34, $\frac{1}{4}$, $\frac{8}{10}$, 8.15
6. 1.98, 0.23, 0, 1.89, $\frac{3}{5}$, 1.02, $\frac{3}{2}$
7. 2.35, 2.54, 2.01
8. 9.3, $\frac{5}{3}$, 9.90, $\frac{8}{11}$, 3.78, 3.9, $\frac{3}{5}$
9. 0.02, 0, 6.98, $\frac{1}{16}$, 2.2, 6.89, 2.01

Stretch
Use reasoning to compare the fractions. Do not use common denominators.
Explain your reasoning.
1. $\frac{13}{3}$ _______ $\frac{17}{4}$
2. $\frac{3}{16}$ _______ $\frac{6}{31}$
3. $\frac{7}{11}$ _______ $\frac{9}{13}$
**Review**

1. In a video game, a character needs to shine a light through two spinning wheels that have holes in them. The first wheel makes a complete rotation in 7 seconds. The second wheel makes a complete rotation in 9 seconds. The holes are lined up at 0 seconds. How many seconds will pass before they are lined up again?

2. Your aunt's club is planning to sell small bags of different types of beads to people who want to make their own bead jewelry. The table below lists the different types of beads and how many they have.

<table>
<thead>
<tr>
<th>Type of Bead</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oval bead</td>
<td>24</td>
</tr>
<tr>
<td>Metal bead</td>
<td>18</td>
</tr>
</tbody>
</table>

   The club wants to divide these beads into bags so that each bag has exactly the same number of oval beads and metal beads. What is the greatest number of bags that they can make so that all of the beads are used and there is the same number of each bead in each bag?

3. Determine each sum or difference.
   a. \( \frac{1}{8} + \frac{2}{3} \)
   b. \( \frac{7}{6} - \frac{6}{7} \)
Did You Get the Part?
Multiplying and Dividing with Fractions

WARM UP
Write the least common multiple (LCM) of the numbers in each pair.

1. 3, 4
2. 2, 4
3. 8, 3
4. 15, 6
5. 14, 7

You have used area models to represent the products and quotients of whole numbers. How can you use area models and a variety of other models to represent products and quotients that involve positive rational numbers?

LEARNING GOALS
• Model and interpret the multiplication of fractions.
• Model, interpret, and compute the quotient of a whole number divided by a fraction or mixed number.
• Interpret the remainder when dividing a whole number by a fraction or mixed number.
Return of the Area Model

Previously, you used an area model to represent products, to determine factors, and to list multiples of given numbers. In the same way that area models represent whole number multiplication, area models can represent fraction multiplication.

WORKED EXAMPLE

The expression $\frac{1}{4} \times \frac{1}{2}$ means to multiply $\frac{1}{4}$ and $\frac{1}{2}$. When you multiply a fraction by a fraction, you are calculating a part of a part. You can represent the product of two fractions using an area model.

Let's consider an area model for $\frac{1}{4} \times \frac{1}{2}$ and what it represents.

To represent $\frac{1}{4}$ along one side of the model, divide the model into four equal parts along the vertical line. Then shade $\frac{1}{4}$.

To represent $\frac{1}{2}$ along the other side, divide the model along the horizontal line into two equal parts. Then shade $\frac{1}{2}$.

$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

The area of the overlapping region is the product of the fractions.

1. Estimate the product $\frac{2}{3} \times \frac{3}{4}$.

2. Represent the product using the area model.
ACTIVITY 2.1

Multiplying with Mixed Numbers

Let’s look at two methods for multiplying mixed numbers.

Dawson is thinking about how to determine $3\frac{2}{3} \times 2\frac{1}{4}$. He is trying to remember a model he used when he learned how to multiply whole numbers.

He multiplied $25 \times 34$ first to remember the method, and then applied the same strategy to multiply the mixed numbers.

<table>
<thead>
<tr>
<th>Dawson</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25 \times 34$</td>
</tr>
<tr>
<td>$20 \times 30 \quad 4$</td>
</tr>
<tr>
<td>$5 \times 150 \quad 20$</td>
</tr>
<tr>
<td>$600 \quad 80$</td>
</tr>
<tr>
<td>$150 \quad 20$</td>
</tr>
</tbody>
</table>

$3\frac{2}{3} \times 2\frac{1}{4}$

<table>
<thead>
<tr>
<th>$3 \frac{2}{3}$</th>
<th>$2 \frac{1}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{4}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$\frac{4}{3}$</td>
<td>$\frac{2}{12}$</td>
</tr>
</tbody>
</table>

1. Describe the model Dawson used to calculated the product of two mixed numbers.

LESSON 2: Did You Get the Part? • M1-85
2. Lezlee’s correct method is shown. Describe how she calculated the product of two mixed numbers.

Lezlee

\[ \frac{2}{3} \times \frac{1}{4} \]

\[ \frac{11}{3} \times \frac{9}{4} = \frac{99}{12} \]

\[ = \frac{33}{4} \]

\[ = 8 \frac{1}{4} \]

3. Which method do you prefer, Lezlee’s or Dawson’s? Why?

The teachers at Riverside Middle School decide to make trail mix for an upcoming field trip. Ms. Hadley shares her new tropical trail mix recipe with the other teachers. She named it Hawaiian Trail Mix Extravaganza. The recipe for 1 batch is shown.

**Hawaiian Trail Mix Extravaganza**

- \( \frac{3}{8} \) cups of macadamia nuts
- \( \frac{2}{4} \) cups of pumpkin seeds
- \( \frac{3}{8} \) cups of dried cherries
- \( \frac{4}{8} \) cups of popped popcorn
- \( \frac{1}{3} \) cups of corn syrup
- \( \frac{2}{3} \) cups of almonds
- \( \frac{1}{3} \) cups of sunflower seeds
- \( \frac{2}{3} \) cups of honey
- \( \frac{4}{2} \) cups of raisins
- \( \frac{2}{3} \) cups of granola

**Feeds 12 People**
4. The sixth grade teachers are each going to make 3 batches of Hawaiian Trail Mix Extravaganza. For each ingredient, first use benchmark fractions to estimate how many cups of each are needed. Then calculate the exact answer. Show your work.

a. almonds Estimate: __________

b. popped popcorn Estimate: __________

c. macadamia nuts Estimate: __________

5. There are more seventh grade students than sixth grade students. The seventh grade teachers determine that they are each going to make 4\(\frac{1}{2}\) batches. For each ingredient, first estimate how many cups of each are needed. Then calculate the exact answer. Show your work.

a. raisins Estimate: __________

b. sunflower seeds Estimate: __________

c. pumpkin seeds Estimate: __________
Division often means to ask how many groups of a certain size are contained in a number.

**WORKED EXAMPLE**

The expression $12 \div 3$ means you are trying to determine how many groups of 3 are in 12. A physical model and number line model are shown.

**Physical Model**

1 group of 12

**Number Line Model**

1 group of 12

4 groups of 3

4 groups of 3

There are 4 groups of 3 in 12.

**WORKED EXAMPLE**

When you divide with fractions, you are asking the same question. The expression $2 \div \frac{1}{2}$ is asking how many halves are in 2.

**Physical Model**

There are four $\frac{1}{2}$ parts in 2, so $2 \div \frac{1}{2} = 4$. 
1. For each problem situation, first estimate the answer. Then draw a diagram and write the appropriate number sentence.

   a. How many students can be served with 4 cups of trail mix if each student gets $\frac{1}{2}$ of a cup of trail mix?

   b. How many $\frac{1}{4}$-cup servings of trail mix can you make with 4 cups?

   c. How many $\frac{1}{3}$-cup trail mix servings can you make with 4 cups?

   d. Do you notice a pattern? Explain your reasoning.

2. You have 4 cups of trail mix. If each student receives:

   a. $\frac{2}{3}$ cup, how many students are there?

   b. $\frac{2}{5}$ cup, how many students are there?

   c. $\frac{4}{5}$ cup, how many students are there?

   d. $\frac{4}{7}$ cup, how many students are there?

   e. What patterns do you notice? Explain your reasoning.
3. Jamilla is throwing a small party. She has 4 pizzas and decides that everyone at her party should receive a serving size that is $\frac{3}{5}$ of a pizza. Jamilla says she has $6\frac{2}{3}$ servings, but Devon says she has $6\frac{2}{5}$ servings. Draw a diagram of the situation, and solve for the quotient to determine who is correct. Then explain why one person is not correct.

TALK the TALK

Reasoning with Division

1. How is the quotient of $12 \div \frac{1}{3}$ related to the quotient of $12 \div \frac{2}{3}$? Explain your reasoning.

2. Determine the quotient for each. Then, describe any patterns that you notice.

$$6 \div \frac{1}{2} \quad 6 \div \frac{1}{4} \quad 6 \div \frac{1}{8} \quad 6 \div \frac{1}{16}$$
**Assignment**

**Write**
Describe a way to estimate the quotient of two fractions or mixed numbers. Provide an example.

**Remember**
Division often means to ask how many groups of a certain size are contained in a number. So, $6 \div \frac{4}{5}$ can mean, "How many groups of $\frac{4}{5}$ are in 6?"

**Practice**
Calculate each product or quotient.
1. $2\frac{2}{3} \times 3\frac{1}{3}$
2. $8 \div \frac{3}{4}$
3. $10 \div \frac{2}{5}$
4. $3\frac{3}{5} \times 2\frac{1}{2}$
5. $1\frac{3}{8} \times 6\frac{1}{4}$
6. $5\frac{2}{3} \times 4\frac{1}{6}$
7. $2\frac{1}{3} \times 7\frac{1}{4}$
8. $5 \div \frac{2}{5}$
9. $9 \div \frac{3}{8}$

**Stretch**
Jennifer is buying school supplies for her 3 children, and they each have their own list.
- **Mia:** 15 pencils, 2 erasers, 4 colored markers
- **Cooper:** 16 pencils, 12 pens, 10 colored markers, and 2 erasers
- **Tyler:** 20 pencils, 10 erasers, and 10 sleeves of stickers

<table>
<thead>
<tr>
<th>Single</th>
<th>Pack</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assorted Colored Markers $0.73 per marker</td>
<td>$4.56 for 8 markers</td>
</tr>
<tr>
<td>Erasers $0.18 per eraser</td>
<td>$0.75 for 6 erasers</td>
</tr>
<tr>
<td>Pencils $0.93 per pencil</td>
<td>$10.45 for 12 pencils</td>
</tr>
<tr>
<td>Assorted Stickers $1.07 per sleeve</td>
<td>$5.27 for 5 sleeves</td>
</tr>
<tr>
<td>Assorted Pens $0.72 per pen</td>
<td>$6.85 for 10 pens</td>
</tr>
</tbody>
</table>

Jennifer has budgeted $75 to spend on supplies. Is this an appropriate amount based on the cost list? Explain your reasoning.
Review
1. A school participates in a reading contest. The table shows each sixth grade class's portion of the grade's total reading minutes. Order the classes from the greatest number of reading minutes to the least. Explain your reasoning.

<table>
<thead>
<tr>
<th>Class</th>
<th>Portion of Reading Minutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mr. Karlie</td>
<td>$\frac{5}{12}$</td>
</tr>
<tr>
<td>Ms. Jacobs</td>
<td>$\frac{1}{18}$</td>
</tr>
<tr>
<td>Ms. Suarez</td>
<td>$\frac{4}{9}$</td>
</tr>
<tr>
<td>Mr. Mitchell</td>
<td>$\frac{1}{12}$</td>
</tr>
</tbody>
</table>

2. Order the fractions $\frac{3}{7}$, $\frac{4}{5}$, $\frac{5}{9}$ and $\frac{1}{8}$ from least to greatest. Explain your method.

3. An artist is weaving a rectangular rug to match the pattern shown in the figure. Calculate the area of the entire rug.

4. You are making a kite out of nylon fabric. Study the diagram. How much nylon fabric will you need to make the kite?

5. Estimate and then calculate each product.
   a. $625 \times 34$
   b. $1014 \times 59$
WARM UP
Use benchmark fractions to estimate each product.
1. $2\frac{5}{6} \times 3\frac{1}{8}$
2. $3\frac{8}{9} \times 2\frac{7}{15}$
3. $9\frac{6}{7} \times 4\frac{1}{5}$
4. $6\frac{4}{7} \times 2\frac{1}{9}$

LEARNING GOALS
- Model the division of fractions using area models and on a number line.
- Compute and interpret quotients of fractions and interpret remainders.
- Divide with mixed numbers.

KEY TERMS
- complex fraction
- reciprocal
- multiplicative inverse
- Multiplicative Inverse Property

You have learned how to multiply and divide with whole numbers and positive rational numbers. How can you apply what you know about operating with these numbers to understand how to divide two fractions or mixed numbers?
Getting Started

All in the Fact Family

Write the multiplication-division fact family for each rectangular array.

1.

2.

3.

4. For each fact family, which numbers represent the side lengths of the area model? Which numbers represent the area?
Collect all the diagrams you sorted in the lesson *Thinking Rationally*. Just like fact families for whole-number area models, you can also write multiplication-division fact families for models involving fractions.

Consider the model shown.

The shaded area represents the fraction $\frac{1}{20}$, because 1 rectangle is shaded of the 20 total unit rectangles.

The height of the shaded rectangle is $\frac{1}{5}$ of the height of the model.

The width of the shaded rectangle is $\frac{1}{4}$ of the width of the model.

So, the shaded area of the rectangle represents the product $\frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$.

1. Write a multiplication-division fact family for the model.

2. Describe how the model shows the division of fractions.

3. Write multiplication-division fact families with fractions for the remaining diagrams that you sorted. Show your work.
You can also use fraction strip models to represent fraction division. For example, this model shows $\frac{3}{4} \div \frac{1}{4}$. The division expression asks, how many $\frac{1}{4}$'s are in $\frac{3}{4}$?

1. What is the quotient: $\frac{3}{4} \div \frac{1}{4} = ?$

2. Write a sentence to describe the answer.

3. Write a sentence to describe what each division expression is asking. Then, draw a fraction-strip diagram to represent the division problem. Finally, calculate the quotient and write a sentence to describe your answer.

   a. $\frac{3}{2} \div \frac{1}{4}$  
   b. $\frac{1}{2} \div \frac{1}{8}$  
   c. $\frac{3}{4} \div \frac{1}{8}$
4. How can you check each of your answers in Question 3 to make sure you were correct? Explain your reasoning.

5. Mason has $\frac{2}{3}$ of a foot of ribbon. He needs to divide the ribbon into $\frac{1}{6}$-foot pieces. How many pieces can he cut from the ribbon? Write a division problem to represent this situation. Use the ruler to answer the question and show your work.

```
<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>6</th>
<th>7</th>
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<td></td>
</tr>
</tbody>
</table>
```

**ACTIVITY 3.3**

**Dividing Across**

In the same way that you can “multiply across,” or multiply the numerators and multiply the denominators, to determine the product of two fractions, you can also “divide across” to determine the quotient of two fractions.

**WORKED EXAMPLE**

Determine the quotient: $\frac{7}{8} \div \frac{1}{2} = ?$

Divide the numerators. Then divide the denominators.

$$\frac{7}{8} \div \frac{1}{2} = \frac{7}{8} \div \frac{1}{2} = \frac{7}{8} \times \frac{2}{1} = \frac{7}{4}$$
Amy and Sandy used different ways to calculate the quotient \( \frac{3}{4} \div \frac{1}{3} \).

**Amy**

\[
\frac{3}{4} \div \frac{1}{3} = \frac{9}{12} \div \frac{4}{12} = \frac{9 \div 4}{1} = \frac{9}{4}
\]

I can determine equivalent fractions and then divide across.

**Sandy**

\[
\frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \div \frac{1}{3} = \frac{3}{4} \times \frac{3}{4} = \frac{9}{4}
\]

I just divide across. If I get a fraction over a fraction, I can make the resulting denominator a 1.
A **complex fraction** is a fraction that has a fraction in either the numerator, the denominator, or both the numerator and denominator.

1. **Study Sandy’s and Amy’s methods.**
   a. Which student wrote complex fractions?
   b. How are the methods different? How are they alike?

2. **Calculate each quotient by dividing across.**
   **Rewrite any improper fractions as mixed numbers.**
   a. \( \frac{3}{4} \div \frac{1}{3} \)
   b. \( \frac{3}{8} \div \frac{1}{4} \)
   c. \( \frac{5}{6} \div \frac{2}{3} \)
   d. \( \frac{7}{8} \div \frac{3}{4} \)

---

**Activity 3.4**

Multiply by the Reciprocal

When you reverse the numbers in the numerator and denominator of a fraction, you form a new fraction called the **reciprocal** of the original fraction.

1. **Which number is its own reciprocal?**

2. **Which number has no reciprocal? Explain your reasoning.**

---

The **reciprocal** of a number is also known as the multiplicative inverse of the number. The **multiplicative inverse** of a number \( \frac{b}{a} \) is the number \( \frac{a}{b} \), where \( a \) and \( b \) are nonzero numbers. The product of any nonzero number and its multiplicative inverse is 1.

The **Multiplicative Inverse Property** states: \( \frac{a}{b} \cdot \frac{b}{a} = 1 \), where \( a \) and \( b \) are nonzero numbers.
3. Alexa wrote the reciprocal of the mixed number incorrectly. Explain why she is incorrect and provide the correct reciprocal.

Alexa
Given $3\frac{5}{8}$
The reciprocal is $3\frac{5}{8}$.

Karen said, “I wish everything could be as easy as dividing by 1.” She tried her “dividing by 1” method to determine the quotient $\frac{5}{8} \div \frac{3}{4}$.

“If I can turn the divisor of $\frac{3}{4}$ into 1, then the problem can be solved. I can multiply both fractions by the reciprocal of $\frac{3}{4}$, which is $\frac{4}{3}$, to create 1.”

4. Analyze Karen’s method for dividing fractions. Describe the steps in the dashed circles.

5. Write a rule based on Karen’s method that you can use to calculate the quotient in a fraction division problem.

6. Calculate each quotient.
   a. $\frac{5}{6} \div \frac{1}{4}$
   b. $\frac{4}{5} \div \frac{1}{3}$
   c. $\frac{1}{8} \div \frac{1}{2}$
   d. $\frac{3}{10} \div \frac{1}{3}$
Let’s consider how to make a bag of trail mix that has a weight greater than 1 pound.

If you have $5\frac{2}{3}$ pounds of trail mix, how many bags can you make so that each bag contains $1\frac{5}{6}$ pounds?

Analyze each student’s method.

**Carla**

I drew a model for $5\frac{2}{3}$.

I knew that I needed $\frac{5}{6}$ groups, so I divided my model to show $\frac{1}{6}$’s. Because $\frac{5}{6} = \frac{11}{6}$, I then marked off groups of $\frac{11}{6}$.

The remaining $\frac{1}{6}$ part is actually $\frac{1}{11}$ of a group.

So, I can make $3\frac{1}{11}$ bags of trail mix.
Karen

I wrote a division sentence, and then converted both mixed numbers to improper fractions.

\[
5 \frac{2}{3} \div 1 \frac{5}{6} = \frac{17}{3} \div \frac{11}{2} \\
= \frac{17}{3} \cdot \frac{2}{11} = \frac{34}{33} \\
= 3 \frac{1}{33}
\]

So, I can make 3\frac{1}{33} bags of trail mix.

1. Karen converted the mixed numbers to improper fractions. How did Carla represent this same step?

2. Describe how Karen changed from division to multiplication.

Solve each problem. Show your work and be sure to label your answer.

3. The cook in the school cafeteria made 47\frac{1}{2} cups of mashed potatoes. If there are 1\frac{1}{4} cups of mashed potatoes in a serving, how many servings did she make?

4. One of the most beautiful hiking trails in the United States is Glacier Gorge in Rocky Mountains National Park. The hiking trail through Glacier Gorge is 9\frac{3}{5} miles round trip. If you hike 1\frac{3}{5} miles an hour, how many hours will the round trip take?
TALK the TALK

Going (Almost) Numberless

1. Complete each statement with greater than, less than, or the same as.

   a. If a quantity greater than 1 is divided by a value between 0 and 1, the quotient will be _____________ the original quantity.

   b. If a quantity between 0 and 1 is divided by a value greater than 1, the quotient will be _____________ the original quantity.

   c. If a quantity between 0 and 1 is divided by a value between 0 and 1, the quotient will be _____________ the original quantity.

2. Complete each statement with always, sometimes, or never.

   a. If a mixed number is divided by another mixed number, the quotient will _____________ be greater than 1.

   b. If a fraction between 0 and 1 is multiplied by another fraction between 0 and 1, the product will _____________ be less than 1.

   c. If a whole number is divided by a fraction between 0 and 1, the quotient will _____________ be less than 1.

   d. If a fraction between 0 and 1 is multiplied by a mixed number, the product will _____________ be greater than 1.
3. Consider the quotients $\frac{5}{6} \div \frac{1}{2}$ and $\frac{5}{6} \div 2$.

   a. Describe how these quotients are different.

   b. Write a real-world problem that can be solved using each division.
**Assignment**

**Write**
Explain how an area model can represent the division of two fractions.

**Remember**
One way to divide two fractions is to divide across:

\[
\frac{3}{4} \div \frac{1}{4} = \frac{3}{4} \div \frac{1}{4} = \frac{3}{1}
\]

Another way is to rewrite the division problem as multiplication by the reciprocal of the divisor:

\[
\frac{3}{4} \div \frac{1}{4} = \frac{3}{4} \times \frac{4}{1} = \frac{12}{4} = 3
\]

**Practice**
Calculate each quotient.

1. \( \frac{2}{3} \div \frac{1}{3} \)
2. \( \frac{7}{8} \div \frac{1}{4} \)
3. \( \frac{3}{4} \div \frac{1}{6} \)
4. \( \frac{15}{16} \div \frac{3}{4} \)
5. \( \frac{7}{12} \div \frac{1}{3} \)
6. \( 1 \frac{1}{8} \div \frac{5}{6} \)
7. \( 5 \frac{3}{8} \div \frac{1}{4} \)
8. \( 7 \frac{1}{3} \div 1 \frac{2}{3} \)

**Stretch**
Write a word problem that could be modeled by the quotient \( 2 \frac{1}{2} \div \frac{3}{4} \).
**Review**

1. A triathlon competition consists of swimming, cycling, and running. Not all races cover the same distances. According to USA Triathlon, the international distance triathlon consists of $\frac{9}{10}$ mile swimming, $24\frac{3}{8}$ miles cycling, and $6\frac{1}{2}$ miles running. One of the most famous triathlons is an Ironman competition. Competitors in an Ironman competition must swim $2\frac{2}{3}$ times farther than competitors in an international distance triathlon.
   a. Use benchmark fractions to estimate how far competitors must swim in an Ironman triathlon. Show your work.
   b. Calculate the exact distance competitors in an Ironman triathlon must swim. Show your work.

2. Ling is a camp counselor at a local summer camp. She is in charge of the weekly craft activity for 40 campers. She plans to make fabric-covered frames that each require $\frac{1}{6}$ yard of fabric. The camp director gave her $6\frac{2}{3}$ yards of fabric remnants for this project. Does Ling have enough fabric for her craft activity? Show your work.

3. Write the prime factorization for each number. Then, determine the greatest common factor.
   a. 28, 32
   b. 40, 100

4. Draw a model to determine each quotient.
   a. $4 \div \frac{5}{4}$
   b. $2 \div \frac{4}{3}$
Positive Rational Numbers Summary

KEY TERMS
- positive rational number
- benchmark fraction
- complex fraction
- reciprocal
- multiplicative inverse
- Multiplicative Inverse Property

A positive rational number is a number that can be written in the form $\frac{a}{b}$, where $a$ and $b$ are both whole numbers greater than 0.

Any decimal greater than 0 that has a limited number of digits after the decimal point (like 0.5) or whose digits repeat in a pattern (like 0.3333 . . .) is a positive rational number.

For example, is 0.75 a rational number?
To write a decimal like 0.75 in the form $\frac{a}{b}$, where $a$ and $b$ are both whole numbers and $b$ is not equal to 0:
- Read the decimal using place value.
  0.75 $\rightarrow$ seventy-five hundredths
- Write the decimal as a fraction.
  0.75 $\rightarrow$ $\frac{75}{100}$
The fraction $\frac{75}{100}$ is written in the form $\frac{a}{b}$, where $a$ is equal to 75 and $b$ is equal to 100. The numbers 75 and 100 are both whole numbers greater than 0.
So, 0.75 is a rational number.
**Benchmark fractions** are common fractions you can use to estimate the value of fractions. Three common benchmark fractions are $\frac{0}{1}$, $\frac{1}{2}$, and $\frac{1}{1}$.

A fraction is close to 0 when the numerator is very small compared to the denominator. A fraction is close to $\frac{1}{2}$ when the numerator is about half the size of the denominator. A fraction is close to 1 when the numerator is very close in size to the denominator.

An inequality is a statement that one number is less than or greater than another number. For example, write an inequality comparing the fractions $\frac{7}{8}$, $\frac{1}{4}$, and $\frac{2}{5}$.

**Using Benchmark Fractions**
- $\frac{7}{8}$ is close to the benchmark $\frac{1}{1}$, or 1.
- $\frac{1}{4}$ is close to the benchmark $\frac{0}{1}$, or 0.
- $\frac{2}{5}$ is close to the benchmark $\frac{1}{2}$.

So, $\frac{1}{4} < \frac{2}{5} < \frac{7}{8}$.

**Using Equivalent Fractions**
- $\frac{7}{8} = \frac{35}{40}$
- $\frac{1}{4} = \frac{10}{40}$
- $\frac{2}{5} = \frac{16}{40}$

**LESSON 2**

**Did You Get the Part?**

You can use area models to multiply mixed numbers or you can write the mixed numbers as improper fractions before multiplying. For example, calculate the product of $3\frac{2}{3} \times 2\frac{1}{4}$.

**Area Model**

$$
\begin{array}{ll}
& 2 \quad 1 \quad 3 \\
3 & 6 \quad \frac{3}{4} \\
3 & \frac{4}{3} \quad \frac{2}{12} \\
\end{array}
$$

**Improper Fractions**

$$
3\frac{2}{3} \times 2\frac{1}{4} = \frac{11}{3} \times \frac{9}{4}
$$

$$
= 99 \div 12 = 33 \div 4 = 8\frac{1}{4}
$$

$$
= 6 + \frac{27}{12} = 6 + \frac{3}{12} = 6 + \frac{1}{4} = 8\frac{1}{4}
$$
Division often means to ask how many groups of a certain size are contained in a number. When you divide with fractions, you are asking the same question. Examine the models shown.

**Physical Model**

![Physical Model Diagram]

The expression $2 \div \frac{1}{2}$ is asking how many halves are in 2.

In another example, the expression $6 \div \frac{4}{5}$ can mean, “How many groups of $\frac{4}{5}$ are in 6?”

There are 7 whole groups of $\frac{4}{5}$ in 6 and what is left over is half of a group of $\frac{4}{5}$. So, $6 \div \frac{4}{5} = \frac{35}{4}$.

**Number Line Model**

![Number Line Model Diagram]

There are four $\frac{1}{2}$ parts in 2, so $2 \div \frac{1}{2} = 4$.

You can write multiplication-division fact families for models involving fractions.

For example, the shaded area represents the fraction $\frac{1}{20}$. The height of the shaded rectangle is $\frac{1}{5}$ the height of the model and the width of the shaded rectangle is $\frac{1}{4}$ the width of the model.

So, the shaded area of the rectangle represents the product $\frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$.

Therefore, $\frac{1}{20} \div \frac{1}{5} = \frac{1}{4}$ and $\frac{1}{20} \div \frac{1}{4} = \frac{1}{5}$.

You can also use fraction strip models to represent fraction division. For example, this model shows $\frac{3}{4} \div \frac{1}{4}$. The division expression asks, how many $\frac{1}{4}$s are in $\frac{3}{4}$?

\[
\frac{3}{4} \div \frac{1}{4} = 3
\]
You can “divide across” to determine the quotient of two fractions.

For example, determine the quotient: \( \frac{7}{8} \div \frac{1}{2} \).

Divide the numerators. Then divide the denominators. \( \frac{7}{8} \div \frac{1}{2} = \frac{7 \div 1}{8 \div 2} = \frac{7}{4} \)

You may sometimes write a complex fraction while dividing across. A complex fraction is a fraction that has a fraction in either the numerator, the denominator, or both the numerator and denominator. You can use the reciprocal of a number to change a complex fraction to a rational number.

The reciprocal of a number is also known as the multiplicative inverse of the number. The multiplicative inverse of a number \( \frac{a}{b} \) is the number \( \frac{b}{a} \), where \( a \) and \( b \) are nonzero numbers.

The Multiplicative Inverse Property states that \( \frac{a}{b} \times \frac{b}{a} = 1 \), where \( a \) and \( b \) are nonzero numbers.

Another way to determine the quotient of two fractions is to multiply by the reciprocal of the divisor.

You can use any of these methods to divide mixed numbers as well.

For example, if you have \( \frac{5\frac{2}{3}}{4} \) pounds of trail mix, how many bags can you make so that each bag contains \( \frac{1\frac{5}{6}}{4} \) pounds? Write a division sentence and then convert both mixed numbers to improper fractions.

\[
\begin{align*}
5\frac{2}{3} \div 1\frac{5}{6} &= \frac{17}{3} \div \frac{11}{6} \\
&= \frac{17}{3} \cdot \frac{6}{11} \\
&= \frac{17}{3} \cdot \frac{2}{11} \\
&= \frac{34}{33} \\
&= 3\frac{1}{33}
\end{align*}
\]

Multiply the dividend by the reciprocal of the divisor.

Write the product as a mixed number.
TOPIC 3

Decimals and Volume

The Louvre Pyramid in Paris was designed by Chinese-American architect I.M. Pei.

Lesson 1
Length, Width, and Depth
Deepening Understanding of Volume ........................................ M1-115

Lesson 2
Which Warehouse?
Volume Composition and Decomposition .............................. M1-131

Lesson 3
Breaking the Fourth Wall
Surface Area of Rectangular Prisms and Pyramids .................... M1-143

Lesson 4
Dividend in the House
Dividing with Volume and Surface Area ................................. M1-165
TOPIC 3: DECIMALS AND VOLUME
This topic builds on students’ prior knowledge of volume, area, and decimal operations. Students are introduced to the language of prisms and pyramids so that distinctions can be made as they solve volume and surface area problems. Through problem-solving activities with volume, students review addition and subtraction of decimal numbers and continue operating with decimals, with the eventual goal of fluency. Students decompose three-dimensional solids into two-dimensional nets and compose solids from nets. Students review whole-number and decimal multiplication and learn how to long divide with whole numbers and decimals.

Where have we been?
Students began learning about decimals in grade 4 and 5. They have experience using concrete models and place-value strategies to operate with decimals to the hundredths place. In grade 5, students learned how to calculate the volume of a right rectangular prism by filling it with cubes and eventually by using the formulas \( V = lwh \) and \( V = Bh \).

Where are we going?
Students will use decimal operations to solve real-world and mathematical problems throughout the remaining modules of this course. Fractions and decimals are encountered more frequently than whole numbers in daily life, so students should be comfortable and confident solving problems that require operating with such numbers.

Using Nets to Construct Models of Solid Figures
A net is a two-dimensional model that can be folded into a three-dimensional solid.

The net shown is the net of a cube, which is a rectangular prism that has 6 square faces that are the same size. The net helps to show the entire surface area of the cube.
Myth: Asking questions means you don’t understand.

It is universally true that, for any given body of knowledge, there are levels to understanding. For example, you might understand the rules of baseball and follow a game without trouble. But there is probably more to the game that you can learn. For example, do you know the 23 ways to get on first base, including the one where the batter strikes out?

Questions don’t always indicate a lack of understanding. Instead, they might allow you to learn even more about a subject that you already understand. Asking questions may also give you an opportunity to ensure that you understand a topic correctly. Finally, questions are extremely important to ask yourself. For example, everyone should be in the habit of asking themselves, “Does that make sense? How would I explain it to a friend?”

#mathmythbusted

Talking Points

You can further support your student’s learning by asking questions about the work they do in class or at home. Your student is becoming fluent with decimal operations and gaining experience with two- and three-dimensional measures such as square and cubic units.

Questions to Ask

• How does this problem look like something you did in class?
• Can you show me the strategy you used to solve this problem? Do you know another way to solve it?
• Does your answer make sense? Why?
• Is there anything you don’t understand? How can you use today’s lesson to help?

Key Terms

polygon
A polygon is a closed figure that is formed by joining three or more line segments at their endpoints.

polyhedron
A polyhedron is a three-dimensional figure that has polygons as faces.

edge
An edge is the intersection of two faces of a three-dimensional figure. An edge will always be a line segment.

vertex
The point where edges of a three-dimensional figure meet is known as a vertex.
LESSON 1:
Length, Width, and Depth
Deepening Understanding of Volume

WARM UP
Determine the least common multiple of the numbers in each pair.

1. 2, 10
2. 3, 8
3. 6, 14
4. 10, 15

LEARNING GOALS
• Determine the volume of right rectangular prisms with fractional edge lengths using unit cubes with unit fractional dimensions.
• Connect the volume formulas \( V = lwh \) and \( V = Bh \) with a unit-cube model of volume for rectangular prisms.
• Apply the formulas \( V = lwh \) and \( V = Bh \) to determine volumes in real-world problems.
• Fluently add, subtract, and multiply multi-digit decimals using the standard algorithms.

KEY TERMS
• point
• line segment
• polygon
• geometric solid
• polyhedron
• face
• edge
• vertex
• right rectangular prism
• cube
• pyramid
• volume

You know about three-dimensional figures such as cubes and other rectangular prisms. You also know how to operate with positive rational numbers. How can you use what you know to calculate measurements of any rectangular prism, even one with fractional edge lengths?

LESSON 1: Length, Width, and Depth  •  M1-115
Common Figures

Cut out the cards found at the end of the lesson. Sort the figures into two or more groups. Name each category and be prepared to share your reasoning.
It is important to speak a common language when studying mathematics.

A word you may have used in the past may actually have a more precise definition when dealing with mathematics. For example, the word *point* has many meanings outside of math. However, the mathematical definition of *point* is a location in space. A mathematical point has no size or shape, but it is often represented by using a dot and is named by a capital letter. A *line segment* is a portion of a line that includes two points and all the points between those two points. Knowing these definitions will help you learn the meanings of other geometric words.

Recall, a *polygon* is a closed figure formed by three or more line segments.

A *geometric solid* is a bounded three-dimensional geometric figure.

A *polyhedron* is a three-dimensional solid figure that is made up of polygons. A *face* is one of the polygons that makes up a polyhedron. An *edge* is the intersection of two faces of a three-dimensional figure. The point where multiple edges meet is known as a *vertex* of a three-dimensional figure.

Let’s revisit the different figures you sorted.

1. **Sort the figures into one of these three categories and explain your reasoning.**

   - Polygon
   - Polyhedron
   - Neither
Three polyhedra are shown.

Figure A is a right rectangular prism. A right rectangular prism is a polyhedron with three pairs of congruent and parallel rectangular faces.

Figure B is an example of a cube, which is a special kind of right rectangular prism. A cube is a polyhedron that has congruent squares as faces.

Figure C is an example of a rectangular pyramid. A pyramid is a polyhedron with one base and the same number of triangular faces as there are sides of the base.

2. Describe the different faces of each polyhedron.

3. Study the right rectangular prism. Identify the three pairs of congruent parallel faces.

4. Study the cube.
   a. Describe the locations of the cube faces you can see and the locations of the faces you cannot see.
   b. What do you know about the length, width, and height of the cube?
   c. Describe how the cube is also an example of a right rectangular prism.

A unit cube is a cube whose sides are all 1 unit long.

When you have more than 1 vertex, you say “vertices.”
5. Compare the numbers of faces, edges, and vertices of the cube and the other right rectangular prism. Write what you notice.

6. Study the rectangular pyramid. How do the faces of the rectangular pyramid differ from the faces of the rectangular prisms?

7. List examples in the real-world objects that are shaped like right rectangular prisms or pyramids.

---

**ACTIVITY 1.2 Volume of Rectangular Prisms**

*Volume* is the amount of space occupied by an object. The volume of an object is measured in cubic units.

The volume of a cube is calculated by multiplying the length times the width times the height.

\[
\text{Volume of a cube} = l \times w \times h
\]

1. Calculate the volume of each cube with the given side length.
   
   a. \( \frac{9}{10} \) centimeter
   
   b. \( 1\frac{1}{3} \) centimeters

2. Suppose a cube has a volume of 27 cubic meters. What are the dimensions of the cube?
To determine the volume of a rectangular prism, you can also pack the prism with cubes. You may have done this in elementary school.

Consider the rectangular prism shown. What do you notice about the side lengths? Can you determine its volume by packing it with cubes?

![Diagram of a rectangular prism with dimensions 3/4 in., 1 1/2 in., and 1 1/2 in.]

**WORKED EXAMPLE**

To determine the volume of the right rectangular prism with dimensions $1 \frac{1}{2} \times \frac{1}{2} \times \frac{3}{4}$, you can fill the prism with cubes. However, the unit cubes that you may have used in elementary school will not work here. Instead, smaller unit cubes with fractional side lengths are required.

Assign a unit fraction to the dimensions of each cube. Use the least common multiple (LCM) of the fraction denominators to determine the unit fraction.

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Length (1 1/2 ÷ 1/4)</th>
<th>Width (1/2 ÷ 1/4)</th>
<th>Height (3/4 ÷ 1/4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Determine the number of cubes needed to pack the prism in each dimension.

<table>
<thead>
<tr>
<th></th>
<th>length</th>
<th>width</th>
<th>height</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$6 \times 2 \times 3 = 36$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Determine the number of cubes that make up the right rectangular prism.

Multiply the number of cubes by the volume of each cube to determine the volume of the right rectangular prism.

The volume of the right rectangular prism is $\frac{9}{16}$ cubic inches.
3. Interpret the worked example.
   a. How was the number of cubes needed to pack the prism in each dimension determined?
   b. Instead of cubes with a width of \( \frac{1}{4} \) inch, suppose you used cubes each with a width of \( \frac{1}{8} \) inch. How does this change the volume of the rectangular prism?

4. Use the method from the worked example to determine the volume of each rectangular prism.
   a. \( 1 \frac{3}{4} \) in. by \( 2 \frac{1}{3} \) in. by \( \frac{1}{2} \) in.
   b. \( 2 \frac{1}{2} \) in. by \( \frac{3}{4} \) in. by \( \frac{3}{8} \) in.
You have calculated the volume of a rectangular prism using the formula $V = lwh$, where $V$ is the volume, $l$ is the length, $w$ is the width, and $h$ is the height. You also know that the area of a rectangle can be calculated using the formula $A = l \cdot w$.

Consider the two formulas:

$$V = l \cdot w \cdot h$$
$$A = l \cdot w$$

If $B$ is used to represent the area of the base of a rectangular prism, then you can rewrite the formula for area: $B = l \cdot w$.

Now consider the two formulas:

$$V = l \cdot w \cdot h$$
$$B = l \cdot w$$

Using both of these formulas, you can rewrite the formula for the volume of a rectangular prism as $V = B \cdot h$, where $V$ represents the volume, $B$ represents the area of the base, and $h$ represents the height.

In order to calculate the volume of various geometric solids you will need to perform multiplication. In this activity, you will calculate the volume of rectangular prisms with decimal side lengths.

Consider the right rectangular prism shown.

It is good practice to estimate before you actually calculate. If you have an estimate, you can use it to decide whether your answer is correct.
To calculate the volume of the prism, first calculate the area of the base, $B$, by multiplying 32.64 meters by 7.3 meters.

Kenny said, “I use estimation to help place the decimal point correctly in the product.”

**WORKED EXAMPLE**

The area of the base is $32.64 \text{ meters} \times 7.3 \text{ meters}$.

He estimates his two numbers.

32.64 is close to 30
7.3 is close to 7
$30 \times 7 = 210$

So he knows his product is close to 210, but larger since he rounded down. Next, he calculates the product of $32.64 \times 7.3$.

\[
\begin{array}{c}
\text{32.64} \\
\times \quad \text{7.3} \\
\hline
\text{9792} \\
\text{228480} \\
\hline
\text{238.272}
\end{array}
\]

Kenny knows the product will be close to but greater than 210, so he must place the decimal point after the 8. The area of the base of the rectangular prism is 238.272 square meters.

1. Calculate the volume of the right rectangular prism.
2. Each number sentence represents the base, \( B \), times height, \( h \), of different rectangular prisms. Complete each number sentence by inserting a decimal point to show the correct volume.

a. \( 53.6 \text{ sq. ft} \times 0.83 \text{ ft} = 44488 \text{ cu. ft} \)

b. \( 7.9 \text{ sq. cm} \times 0.6 \text{ cm} = 474 \text{ cu. cm} \)

c. \( 0.94 \text{ sq. m} \times 24.9 \text{ m} = 23406 \text{ cu. m} \)

3. Casey thought that using a pattern would help her understand how to calculate the product in a decimal multiplication problem.

a. Complete the table.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Product</th>
<th>Problem</th>
<th>Product</th>
<th>Problem</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 32 \times 100 )</td>
<td>( 3.2 \times 100 )</td>
<td>( 0.32 \times 100 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 32 \times 10 )</td>
<td>( 3.2 \times 10 )</td>
<td>( 0.32 \times 10 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 32 \times 1 )</td>
<td>( 3.2 \times 1 )</td>
<td>( 0.32 \times 1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 32 \times 0.1 )</td>
<td>( 3.2 \times 0.1 )</td>
<td>( 0.32 \times 0.1 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 32 \times 0.01 )</td>
<td>( 3.2 \times 0.01 )</td>
<td>( 0.32 \times 0.01 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 32 \times 0.001 )</td>
<td>( 3.2 \times 0.001 )</td>
<td>( 0.32 \times 0.001 )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Describe any patterns that you notice.
4. A rectangular prism with \( B = 26 \) square centimeters and \( h = 31 \) centimeters has a volume of 806 cubic centimeters. Use this information to determine the volume of the other rectangular prisms.

a. 2.6 sq. cm \( \times \) 31 cm  
b. 2.6 sq. cm \( \times \) 3.1 cm  
c. 0.26 sq. cm \( \times \) 3.1 cm  
d. 2.6 sq. cm \( \times \) 0.31 cm  
e. 0.26 sq. cm \( \times \) 31 cm  
f. 2.6 sq. cm \( \times \) 0.031 cm  
g. 0.026 sq. cm \( \times \) 0.31 cm  
h. 0.26 sq. cm \( \times \) 0.31 cm

5. Look at the patterns in Question 4.

a. How can some of the rectangular prisms have the same volume?

b. How can you tell without multiplying which rectangular prisms will have the same volume?
Fractionally Full

1. Determine the volume of a right rectangular prism with dimensions $1 \frac{1}{4}$ feet $\times$ 1 foot $\times$ $\frac{1}{2}$ foot using the unit fraction method you learned in this lesson.

2. Haley makes earrings and packages them into cube boxes that measure $\frac{1}{6}$-foot wide. How many $\frac{1}{6}$-foot cubic boxes can she fit into a shipping box that is $1 \frac{1}{6}$ feet by $\frac{1}{3}$ foot by $\frac{1}{3}$ foot?

3. The school athletic director has a storage closet that is $4 \frac{1}{2}$ feet long, $2 \frac{2}{3}$ feet deep, and 6 feet tall.
   a. She wants to put carpet in the closet. How much carpeting will she need?
   b. The athletic director wants to store cube boxes that are $\frac{1}{2}$ foot wide. How many boxes will the storage closet hold?

4. Estimate the volume of each right rectangular prism. Then calculate its volume.
   a. 0.1 ft
      1.9 ft
      14.1 ft
   b. 2.5 ft
      4.2 ft
      9.3 ft

M1-126 • TOPIC 3: Decimals and Volume
Assignment

Write
Suppose a rectangular prism has fractional edge lengths. Describe how you can determine the dimensions of cubes that will fill the rectangular prism completely with no overlaps or gaps.

Remember
The volume of a rectangular prism is a product of its length, width, and height: \( V = l \cdot w \cdot h \).

Practice
1. Consider the right rectangular prism shown.

```
<table>
<thead>
<tr>
<th>5.75 cm</th>
<th>2.25 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.5 cm</td>
<td></td>
</tr>
</tbody>
</table>
```

a. List the numbers of faces, edges, and vertices of the rectangular prism.
b. Estimate the volume of the rectangular prism.
c. Calculate the volume of the rectangular prism.

2. Calculate the volume of the rectangular prism with each set of given dimensions.
   a. 7 in. \( \times \) 4 in. \( \times \) 2 in.
   b. 5.2 cm \( \times \) 5.2 cm \( \times \) 12 cm
   c. 11.3 cm \( \times \) 3.5 cm \( \times \) 10.1 cm
   d. 4.5 m \( \times \) 9 m \( \times \) 6.7 m
   e. 2.2 ft \( \times \) 5.5 ft \( \times \) 15 ft

Stretch
Calculate the volume for the triangular prism.

```
\begin{array}{c}
\text{6 cm} \\
\text{5.2 cm} \\
\text{10 cm}
\end{array}
```
Review

1. Elena wants to put together some of her favorite songs on her computer. She wants to store 60 minutes worth of music. Elena wonders how many songs she will be able to include. She looks online and finds a source that says the average song length is $3\frac{1}{2}$ minutes. If this is true, about how many songs will Elena be able to store? Show your work.

2. Ling is a camp counselor at a local summer camp. She is in charge of the weekly craft activity for 40 campers. Ling plans to make fabric-covered frames that each require $\frac{1}{6}$ yard of fabric. When Ling sets up for her craft activity, she measures the four separate fabric remnants her director gave her. The table shows how much of each fabric she has. How many campers can use plaid fabric? Show your work.

<table>
<thead>
<tr>
<th>Fabric</th>
<th>Amount (yards)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plaid</td>
<td>$\frac{11}{12}$</td>
</tr>
<tr>
<td>Tie-dyed</td>
<td>$\frac{7}{9}$</td>
</tr>
<tr>
<td>Striped</td>
<td>$\frac{2}{9}$</td>
</tr>
<tr>
<td>Polka-dotted</td>
<td>$\frac{3}{4}$</td>
</tr>
</tbody>
</table>

3. Represent each product using an area model. Then calculate the product.
   a. $\frac{3}{4} \times \frac{1}{3}$
   b. $\frac{1}{2} \times \frac{3}{5}$

4. Determine the GCF of each set of numbers.
   a. 72 and 30
   b. 30 and 54

5. Determine the LCM of each set of numbers.
   a. 10 and 12
   b. 8 and 9
LESSON 2: Which Warehouse?   •   M1-131

LEARNING GOALS
• Fluently add, subtract, and multiply multi-digit decimals using the standard algorithms.
• Determine volumes of figures composed of rectangular prisms.

KEY TERMS
• composite solid
• trailing zeros

You have calculated areas by composing or decomposing complex shapes into familiar shapes. How can you used this same idea to determine the volume of composite solids?

WARM UP
Calculate each product.
1. 0.5 × 0.5  
2. 0.1 × 0.9
3. 0.3 × 0.9  
4. 0.8 × 0.7
5. 0.7 × 0.7  
6. 0.4 × 0.4
7. 0.6 × 0.7  
8. 0.6 × 0.8
9. 0.3 × 0.2  
10. 0.2 × 0.8

LESSON 2: Which Warehouse?   •   M1-131
Measuring Water

You have two empty containers, each with a different volume, as shown. You also have a source of water.

1. Using just these containers, how can you measure out a volume of exactly 4 gallons (924 in.³)?
As part of Let’s Build Together, an organization that builds recreation centers for communities in need, your class is building a concrete bench for use in a community garden.

Your class has been provided with a drawing of your assignment. You need to determine how much concrete is needed to construct the bench.

The bench is a composite solid. A composite solid is made up of more than one geometric solid.

1. How might you determine the amount of concrete needed to construct your group’s bench? What information do you need to know?

Sofia and Hunter propose different strategies for determining the volume of the bench. Sofia’s strategy requires the addition of volumes and Hunter’s strategy requires the subtraction of volumes. Because of the decimal side lengths of this bench, let’s start by reviewing how to add and subtract with decimals.
Let’s consider adding decimals.

**WORKED EXAMPLE**

\[ 3.421 + 9.5 + 12.85 = ? \]

Before calculating the sum, estimate the answer so you know the approximate sum.

\[ 3 + 10 + 13 = 26 \]

To calculate the exact sum, line up the decimals so that like place values are in the same column. You can use the decimal point as a reference point to help you align numbers in the correct place-value column.

\[
\begin{array}{c}
3.421 \\
9.5 \\
+12.85 \\
\hline
25.771
\end{array}
\]

The estimate of 26 and the sum of 25.771 are reasonably close, so the sum appears to be correct.

2. Lijo says that he can write 9.5 as 9.500 to help calculate the sum \(3.421 + 9.5 + 12.85\).

   a. How does this help Lijo calculate the sum?

   b. How might Lijo rewrite 12.85 in this sum?

3. Summarize how to add decimals.

---

Lijo added trailing zeros to his decimal numbers. **Trailing zeros** are a sequence of 0s in a decimal representation of a number, after which no non-zero digits follow. Trailing zeros do not affect the value of a number.

Estimating first helps you check your answers. You know what answer to expect.
You can use a similar algorithm for subtracting decimals. Let's consider two different subtraction problems.

**WORKED EXAMPLE**

<table>
<thead>
<tr>
<th></th>
<th>18.205 − 3.91</th>
<th>22.4 − 8.936</th>
</tr>
</thead>
<tbody>
<tr>
<td>First, estimate the</td>
<td>18 − 4 = 14</td>
<td>22 − 9 = 13</td>
</tr>
<tr>
<td>answer so you know</td>
<td></td>
<td></td>
</tr>
<tr>
<td>the approximate</td>
<td></td>
<td></td>
</tr>
<tr>
<td>difference.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Then, line up the</td>
<td>7.1110</td>
<td>11.13910</td>
</tr>
<tr>
<td>decimals so that like</td>
<td>18.205</td>
<td>22.40000</td>
</tr>
<tr>
<td>place values are in</td>
<td>−3.910</td>
<td>−8.936</td>
</tr>
<tr>
<td>the same column and</td>
<td>14.295</td>
<td>13.464</td>
</tr>
<tr>
<td>subtract.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Compare the answer</td>
<td>The estimate of</td>
<td>The estimate of</td>
</tr>
<tr>
<td>to your estimate to</td>
<td>14 and the</td>
<td>13 and the</td>
</tr>
<tr>
<td>check your work.</td>
<td>difference of</td>
<td>difference of</td>
</tr>
<tr>
<td></td>
<td>14.295 are</td>
<td>13.464 are</td>
</tr>
<tr>
<td></td>
<td>reasonably</td>
<td>reasonably</td>
</tr>
<tr>
<td></td>
<td>close, so the</td>
<td>close, so the</td>
</tr>
<tr>
<td></td>
<td>difference</td>
<td>difference</td>
</tr>
<tr>
<td></td>
<td>appears to</td>
<td>appears to</td>
</tr>
<tr>
<td></td>
<td>be correct.</td>
<td>be correct.</td>
</tr>
</tbody>
</table>

4. Analyze both subtraction problems.
   a. What do the subtraction problems have in common?

4. Analyze both subtraction problems.
   b. What is different about the subtraction problems in the worked example?

5. Summarize how to subtract decimals.
Let’s go back to determining the amount of concrete needed for your group’s bench.

6. Sofia proposes that the class decompose the bench into rectangular prisms, calculating the volume of each prism, and then adding up the volumes. Use Sofia’s strategy to determine the volume of the bench.

7. Hunter proposes that the class first calculate the total volume of a 1.2 meter cube. Then, they can subtract out the portion of the cube that forms the seat of the bench. Determine the volume of the bench using Hunter’s strategy.

Remember,
volume is measured in cubic units.
8. Compare the volume calculated using Sofia's strategy with the volume calculated using Hunter's strategy.

9. How are Sofia’s and Hunter’s strategies for determining the volume of composite solids like the strategies used to determine the area of composite figures?

---

**Activity 2.2: Fluency with Decimal Operations**

You have seen that you can add, subtract, and of course multiply positive rational numbers, like decimals, to determine volumes. Let's apply what you know to solve problems.

1. Determine the volume of the figure.
2. Regina is building a hot tub next to her swimming pool. The interior dimensions are 6 feet by 7.5 feet. It includes solid bench seating on all four sides. The bench has a width of 1.5 feet. The bench is positioned 1.75 feet from the ground and 2.25 feet from the top as shown.

a. When the hot tub is filled, the water level will be 0.25 feet from the top. How much water will it take to fill the hot tub?

b. How many cubic feet of concrete is needed to build the hot tub?
3. Calculate the volume of each figure. Show your work.

a.  

b.  

0.6 cm  

0.4 cm  

0.4 cm  

0.4 cm  

1 cm
The Volume Warehouse

A business is shopping for warehouse space. Two of their choices are shown.

Warehouse A

The total cost each month for space in Warehouse A is $0.25 times the number of cubic feet used. The total cost each month for space in Warehouse B is $0.15 times the number of cubic feet used.

1. Which warehouse space would you recommend? What information would you need to make this decision? Write your findings in a report to your Director of Finance.
Assignment

Write
Explain how you can estimate the sum or difference of two or more decimals.

Practice
1. Estimate each sum or difference to the nearest whole number. Then, calculate the exact sum or difference.
   a. $4.78 + 67.13 + 3.83$
   b. $5.8 + 7.009 + 45.2$
   c. $56.02 - 3.76 - 15.27$
   d. $25.91 - 12.72 - 0.97$

2. Subtract to determine the volume of the figure.

3. Add to determine the volume of the figure.

Stretch
Calculate the volume of the right prism with the given base.

Remember
You can add and subtract decimals the same way you add and subtract whole numbers. Line up the decimal points and then add or subtract.

\[
\begin{align*}
3.421 + 9.5 + 12.85 &= 25.771 \\
\end{align*}
\]
Review

   a. Calculate the volume of one of the tiny cubes making up the Rubik’s Cube. Show your work.
   b. Calculate the volume of the Rubik’s Cube using your answer to Question 1. Then calculate the volume using the volume formula. Show your work.

2. Ms. Hendrix said that when she was a girl she used to make mixed cassette tapes with her favorite songs. One side of Ms. Hendrix’s cassette tapes had $22\frac{1}{2}$ minutes of available space.
   a. How many $4\frac{2}{5}$-minute songs could Ms. Hendrix record on one side of a cassette tape? Show your work.
   b. Use estimation to help explain how you know your answer to Question 3 is reasonable.

3. Calculate each product.
   a. $\frac{2}{3} \times \frac{4}{9}$
   b. $\frac{1}{6} \times \frac{12}{13}$
WARM UP

Calculate the area of each composite figure.

1. 

   \[ \text{6 yards} \]
   \[ \text{4 yards} \]
   \[ \text{2 yards} \]
   \[ \text{10 yards} \]

2. 

   \[ \text{5 in.} \]
   \[ \text{8 in.} \]
   \[ \text{14 in.} \]
   \[ \text{14 in.} \]
   \[ \text{6 in.} \]
   \[ \text{6 in.} \]
   \[ \text{5 in.} \]
   \[ \text{5 in.} \]

LEARNING GOALS

- Represent solid figures using two-dimensional nets made up of rectangles and triangles.
- Use nets of solid figures to determine the surface areas of the figures.
- Solve real-world and mathematical problems involving surface area.
- Fluently multiply and divide multi-digit decimals using standard algorithms.

KEY TERMS

- net
- surface area
- pyramid
- slant height

You know how to determine how many cubic units fill a rectangular prism. How can you calculate the number of square units it takes to cover the outside of a prism?
Break Down a Cube

A **net** is a two-dimensional representation of a three-dimensional geometric figure. A net is cut out, folded, and taped to create a model of a geometric solid.

1. Cut, fold, and tape the cube net found at the end of the lesson.

2. Are there other nets that form a cube? Circle the other 11 cutouts that can form a cube.

![Cube Net Diagrams]

3. How did you determine which are nets of cubes?

4. What do all of the nets for a cube have in common? Consider the number of faces, edges, and vertices in your explanation.
A net has all these properties:

- The net is cut out as a single piece.
- All of the faces of the geometric solid are represented in the net.
- The faces of the geometric solid are drawn such that they share common edges.

The **surface area** of a polyhedron is the total area of all its two-dimensional faces.

Consider the cube you created.

1. **How is the area of a face of a cube measured?** Analyze the two responses and explain why Leticia is incorrect in her reasoning.

   **Leticia**
   
   This is a 3D figure, which means that its measurements are cubic units.

   **Isaiah**
   
   Surface area is still measuring area, which is always measured in square units.
2. Describe a strategy that you can use to determine the surface area of a cube.

3. Consider the cube net shown. Calculate the surface area.

4. What is the surface area of a unit cube?

5. Let's consider a different rectangular prism.
   a. Use the net to estimate the surface area of the right rectangular prism.

   b. Calculate the surface area of the right rectangular prism. Explain your calculation.
6. Calculate the surface area of the solid figure represented by each net.

a. 

![Net A Diagram]

5.1 ft  2.9 ft  3.5 ft

b. 

![Net B Diagram]

1.9 m  0.5 m

7. Draw a net to represent each solid figure. Label each net with measurements and then calculate the surface area of the solid figure.

a. 

![Solid A Diagram]

\(\frac{3}{4} \text{ m} \quad \frac{3}{8} \text{ m} \quad \frac{5}{8} \text{ m}\)

b. 

![Solid B Diagram]

1.4 in.  1.0 in.  2.03 in.
The base of a prism does not have to be rectangular. The base of a prism can be a triangle, pentagon, hexagon, and so on.

A pyramid is a polyhedron with one base and the same number of triangular faces as there are sides of the base. The vertex of a pyramid is the point at which all the triangular faces intersect.

### Prisms and Pyramids

A **slant height** of a pyramid is the distance measured along a triangular face from the vertex of the pyramid to the midpoint of an edge of the base.

1. **Analyze the figures shown. Then complete the table using the figures.**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Is it a Prism or Pyramid?</th>
<th>Shape of Base</th>
<th>Number of Faces</th>
<th>Number of Vertices</th>
<th>Number of Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Write the names of Figures A, B, C, and D from your completed table.

3. Label each net with the name of the solid it forms.

   a. 
   b. 
   c. 
   d.
1. Locate the nets for the triangular prism and triangular pyramid at the end of the lesson.
   a. Measure the edge lengths of each net with a centimeter ruler. Label the lengths.
   b. Calculate the surface area of each solid figure.
   c. Cut out, fold, and tape each net.
   d. Name each solid.

2. Calculate the surface area of the solid figure represented by each net.
   a. Before folding the net, can you guess what the solid is going to look like?
3. Draw a net to represent each solid figure. Label each net with measurements and then calculate the surface area of the solid figure.

a.

b. The slant heights are all equal. The height of the base is 5.2 cm.

c.
Scents-R-Us produces candles in a variety of shapes. To produce each candle, the company first creates a mold, and then pours hot wax into the mold. When the hot wax cools and solidifies, the mold is removed.
1. Classify the shape of each candle based on the candle mold.
   a. Candle Mold A
   b. Candle Mold B
   c. Candle Mold C
   d. Candle Mold D

2. Use each candle mold to answer each question.
   a. Calculate the surface area of each candle.
   b. How could Scents-R-Us use the surface area of the candles to determine how to price each candle?
NOTES

TALK the TALK

Volume or Area?

1. A rectangular prism has a height of 6 feet, a length of 7.5 feet, and a width of 5 feet.
   a. Draw a net of the rectangular prism and label its measurements.
   b. Calculate the surface area of the prism.

2. Consider the net of the triangular pyramid shown. The net is composed of 4 equilateral triangles, each with a side length of 4 meters and a height of approximately 3.5 meters.
   a. Label the pyramid with its measurements.
   b. Calculate the surface area of the pyramid.

3. Explain in your own words how to determine the surface area of a pyramid.
Cube Net
Triangular Prism Net
Triangular Pyramid Net
Assignment

Write
Match each definition to its corresponding term.

1. The amount of space occupied by an object
   a. volume
2. A regular polyhedron whose six faces are congruent squares
   b. polyhedron
3. The total area of the two-dimensional surfaces that make up a three-dimensional object
   c. cube
4. The distance across a circle through its center
   d. unit cube
5. A geometric solid that is made up of polygons
   e. surface area
6. The intersection of two faces of a polyhedron
   f. volume
7. A closed figure formed by three or more line segments
   g. point
8. A two-dimensional representation of a three-dimensional geometric figure
   h. line segment
9. A cube that is one unit in length, one unit in width, and one unit in height
   i. geometric solid
10. A bounded three-dimensional geometric figure
    j. faces of a polyhedron
11. A portion of a line that includes two points and all the points in between those two points
    k. edge of a polyhedron
12. The polygons that make up a polyhedron
    l. vertex of a polyhedron
13. A location in space
    m. net
14. The point where the edges of a polyhedron meet
    n. diameter

Remember
The surface area of a polyhedron is the sum of all the areas of the faces of the polyhedron.
**Practice**

1. Name the solid figure formed by each net.
   - a. 
   - b. 

2. Draw a net that will form each solid figure.
   - a. 
   - b. 

3. Calculate the surface area of the cube.

4. The pyramid shown has a square base and congruent triangular faces. Calculate the surface area of the pyramid.

5. Estimate and then calculate the surface area of a rectangular prism with a length of 9.06 ft, a width of 4.11 ft, and a height of 6.2 ft.

**Stretch**

A pentagonal prism has pentagons as bases. Each base can be divided into 5 congruent triangles. Determine the surface area of this pentagonal prism.
1. Kendra is a huge fan of Broadway musicals. She wants to record some of her favorite shows. The table shows the musicals she has chosen and how much space they each take up in megabytes.

<table>
<thead>
<tr>
<th>Musicals</th>
<th>Disc Space (megabytes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annie</td>
<td>109.785</td>
</tr>
<tr>
<td>Beauty and the Beast</td>
<td>131.642</td>
</tr>
<tr>
<td>Into the Woods</td>
<td>79.4</td>
</tr>
<tr>
<td>Les Miserables</td>
<td>192.27</td>
</tr>
<tr>
<td>Shrek – The Musical</td>
<td>117.005</td>
</tr>
</tbody>
</table>

a. Calculate the total amount of memory the musicals will take up. Show your work.
b. How much out of 700 megabytes will Kendra have left after she records her musicals? First estimate, then calculate the answer. Show your work.

2. Determine a rational number between the rational numbers in each pair.
   a. Adult male polar bears measure 2.5 to 3 meters tall.

   ![Number Line](2-2.5-3)

   b. The weights of newborn polar bear cubs range from \(\frac{3}{8}\) to \(\frac{1}{2}\) kilogram.

   ![Number Line](0-\frac{3}{8}-\frac{1}{2}-1)

3. Determine each sum.
   a. 1.009 + 6.965
   b. 3.25 + 0.003
Dividend in the House
Dividing with Volume and Surface Area

WARM UP
Estimate each quotient.

1. 268 ÷ 5
2. 1181 ÷ 23
3. 844 ÷ 11
4. 2883 ÷ 46

LEARNING GOALS
• Fluently divide multi-digit whole numbers and decimals using long division.
• Solve real-world problems involving volume.
• Solve real-world problems involving surface area.

In elementary school, you learned strategies to divide two whole numbers. In this course, you learned how to divide fractions by fractions. How can you use a standard algorithm for dividing whole numbers and decimals to solve problems?
Dimensions of a Tank

The Think Tank designs and creates customized tanks and aquariums for oceanographers. A team of oceanographers who study the characteristics of plankton requested several tanks that have a volume of 240 cubic feet and bases with various areas, but they didn't give any heights. Provide The Think Tank with tank heights using the information given.

1. \( B = 10 \) square feet

2. \( B = 15 \) square feet

3. \( B = \frac{46}{3} \) square feet
As you demonstrated, if you know the volume of a right rectangular prism and the area of the base you can divide to determine the height. Likewise, if you know the volume and the height of a rectangular prism, you can calculate the area of the base. If the volume of a right rectangular prism is 3.57 cubic feet and the height is 3 feet, what strategy can you use to determine the area of the base?

You can use hundredths grids to model dividing decimals.

**WORKED EXAMPLE**

Let’s consider $3.57 \div 3$.

First, represent 3.57. Shade 3 hundredths grids to represent 3. Shade 5 columns in a fourth grid to represent 5 tenths. Then shade 7 more squares to represent 7 hundredths.

Next, divide the shaded model into 3 equal groups. To do this, divide the 3 hundredths grids into 3 equal groups. Then, divide the 57 hundredths into 3 equal groups.

One whole grid and 19 small squares are in each group. So, $3.57 \div 3 = 1.19$. Therefore, the area of the base of a rectangular prism with a volume of 3.57 cubic feet and a height of 3 feet is 1.19 square feet.
1. Compare the two worked examples.

a. What is the area of the base of the right rectangular prism?

b. Describe how the hundredths grid model represents different parts of the standard algorithm.

c. Why does the standard algorithm show subtracting 3 from the 3 ones in the dividend?
d. What does the 05 represent in the standard algorithm?

e. What does 27 – 27 represent in the standard algorithm? Use the hundredths grid model to help you explain.

2. The volume of a right rectangular prism is 26,112 cubic feet and its base has an area of 256 square feet. What is the height? Examine each solution. What did Dustin do incorrectly?

Morgan
I used my strategy from earlier.

\[
\begin{array}{c}
102 \\
256)26,112 \\
-256 \\
512 \\
-512 \\
\end{array}
\]

The height of the right rectangular prism is 102 feet.

Dustin
The height of the prism should be 12 feet.

\[
\begin{array}{c}
12 \\
256)26,112 \\
-256 \\
512 \\
-512 \\
\end{array}
\]

3. The area of the base of the rectangular prism is 1.19 square feet. Calculate the width of each rectangular prism with the given length.

a. Length = 2 feet       b. Length = 3 feet       c. Length = 4 feet

LESSON 4: Dividend in the House • M1-169
You have seen how to divide decimals by whole numbers. Let’s think about how to divide decimals by decimals.

1. Look at these division problems.

\[
\begin{align*}
7 \div 56 & \quad 70 \div 560 & \quad 700 \div 5600 & \quad 7000 \div 56000
\end{align*}
\]

a. How are the divisors and dividends in the last three problems related to the first problem?

b. Calculate all four quotients. What do you notice about them?

c. What happens to the quotient when the dividend and divisor are multiplied by the same number?

2. Which of the division expressions shown have the same quotient as \(475 \div 25\)? How do you know?

\[
\begin{align*}
a. \quad 4.75 \div 0.25 & \quad & b. \quad 47.5 \div 0.025 \\
c. \quad 0.475 \div 0.25 & \quad & d. \quad 0.0475 \div 0.0025
\end{align*}
\]
Let's investigate an algorithm for dividing a decimal by a decimal.

You already know how to divide a decimal by a whole number. You also know that if you multiply or divide both the dividend and the divisor by the same number, the quotient remains the same.

**WORKED EXAMPLE**

The diagram shows 7.7 ÷ 3.5.

The first step is to rewrite the division sentence so the divisor is a whole number. Multiply both the divisor and dividend by 10. This changes the value of both numbers.

77 divided by 35 is 2 with 7 left over, because
\[2 \times 35 + 7 = 77.\]

70 tenths divided by 35 is 2 tenths with 0 left over.

Place the decimal point in the quotient.

3. Examine the worked example.

a. Explain how the worked example shows that \( \frac{7.7}{3.5} = \frac{77}{35} \).
b. Why does the diagram show subtracting 70 from the 77 in the dividend?

c. What does 70 – 70 represent in the diagram?

4. Rewrite each division sentence so the divisor is a whole number. Then calculate the quotient using long division.

   a. \( \frac{59.5}{0.1} \)  b. \( \frac{0.0145}{0.5} \)  c. \( \frac{19.36}{3.2} \)
Let's apply what you have learned about decimal operations to solve problems with volume and surface area.

1. The surface area for a cube is given. Calculate the area of each face of the cube.
   
   a. 36.45 square inches  
   b. 768 square feet  
   c. 59.94 square centimeters

2. Marjorie uses a loaf pan to make cornbread. The pan is 8.5 inches long, 4.5 inches wide, and 2.5 inches deep.
   
   a. The pan has a volume of approximately 6.6 cups. What is the approximate volume of each cup in cubic inches? Estimate and then calculate your answer. Show your work.
   
   b. The cornbread Marjorie makes fills only half the depth of the loaf pan. How much cornbread does Marjorie make? Give your answer in cups and cubic inches.
**TALK the TALK**

**A Short Long Division Activity**

Use the standard algorithm to determine each quotient.

1. \( 5168 \div 646 \)
2. \( 331.25 \div 53 \)

3. \( 9.44 \div 2.95 \)
4. \( 6.85 \div 0.5 \)

For each division statement given, write two division statements that have the same quotient.

5. \( 50.32 \div 7.4 = 6.8 \)
6. \( 10.4 \div 2.6 = 4 \)
Assignment

Write
Describe what is meant by the operation of division.

Remember
In a division sentence, if you multiply the dividend and the divisor by the same number, the quotient remains the same.

\[
7.7 \div 3.5 = 2.2
\]

\[
77 \div 35 = 2.2
\]

\[
\frac{77}{35} = \frac{77}{35}
\]

Practice
Estimate each quotient. Then calculate the quotient using long division. Round to the nearest hundredth.

1. \[51.68 \div 8\]
2. \[93.45 \div 6.23\]
3. \[29.988 \div 2.04\]
4. \[38 \div 7\]
5. \[49.7 \div 25.3\]
6. \[118 \div 26\]
7. \[24.4 \div 8.3\]
8. \[8.603 \div 98\]

Stretch
The volume of the trapezoidal prism is 1279.152 cubic feet. Determine the height of the trapezoid base.

Review
1. Mary Alice has decided to give her best friend a candle for her birthday. To wrap the candle, she spends $2.50 on a rectangular sheet of wrapping paper that is 24 inches by 19.5 inches. How many square inches are in one rectangular sheet of wrapping paper?
2. Calculate the surface area of a Rubik’s Cube that has a width of 57 millimeters.
3. Determine the area of a triangle that has a height of 4 feet and a base of \(6\frac{1}{2}\) feet.
4. Determine the quotient.
   \[12\frac{3}{4} \div 1\frac{1}{5}\]
5. Determine the product of each.
   a. \[3.01 \times 5.8\]  
   b. \[1.2 \times 1.2\]
Decimals and Volume Summary

KEY TERMS
- point
- line segment
- polygon
- geometric solid
- polyhedron
- face
- edge
- vertex
- right rectangular prism
- cube
- pyramid
- volume
- composite solids
- trailing zeros
- net
- surface area
- pyramid
- slant height

LESSON 1
Length, Width, Depth

The mathematical definition of point is a location in space, often represented using a dot and named by a capital letter. A line segment is a portion of a line that includes two points and the points between those two points.

A polygon is a closed figure formed by three or more line segments. A geometric solid is a bounded three-dimensional geometric figure. A polyhedron is a three-dimensional solid figure that is made up of polygons that are called faces. An edge is the intersection of two faces and a vertex is the point where the edges meet.

For example, Figure A is a right rectangular prism, which is a polyhedron with three pairs of congruent and parallel faces.
Figure B is a **cube**, which is a polyhedron that has six congruent squares as faces.

Figure C is a rectangular pyramid. A **pyramid** is a polyhedron with one base and the same number of triangular faces as there are sides of the base.

**Volume** is the amount of space occupied by an object. The volume of an object is measured in cubic units. A unit cube is a cube whose sides are all 1 unit long.

The volume of a rectangular prism is a product of its length, width and height: \( V = l \cdot w \cdot h \).

For example, to determine the volume of the right rectangular prism shown with the given dimensions, you can fill the prism with cubes, but smaller unit cubes with fractional side lengths are required.
Assign a unit fraction to the dimensions of each cube. Use the least common multiple (LCM) of the fraction denominators to determine the unit fraction.

<table>
<thead>
<tr>
<th>LCM(2, 4) = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>So, each cube will measure ( \frac{1}{4} \text{ in.} \times \frac{1}{4} \text{ in.} \times \frac{1}{4} \text{ in.} )</td>
</tr>
<tr>
<td>The volume of each unit cube is ( \frac{1}{64} \text{ cubic inches.} )</td>
</tr>
</tbody>
</table>

Determine the number of cubes needed to pack the prism in each dimension.

<table>
<thead>
<tr>
<th>length</th>
<th>width</th>
<th>height</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \frac{1}{2} \div \frac{1}{4} = 6 )</td>
<td>( \frac{1}{2} \div \frac{1}{4} = 2 )</td>
<td>( \frac{3}{4} \div \frac{1}{4} = 3 )</td>
</tr>
</tbody>
</table>

Determine the number of cubes that make up the right rectangular prism.

\[ 6 \times 2 \times 3 = 36 \]

Multiply the number of cubes by the volume of each cube to determine the volume of the right rectangular prism.

\[ 36 \times \frac{1}{64} = \frac{36}{64} = \frac{9}{16} \]

The volume of the right rectangular prism is \( \frac{9}{16} \) cubic inches.

You can use the formula \( V = Bh \) to calculate the volume of any prism. However, the formula for calculating the value of \( B \) will change depending on the shape of the base. In a rectangular prism, \( B = l \cdot w \).

**Which Warehouse?**

A composite solid is made up of more than one geometric solid. You can decompose a composite solid into more than one polyhedron in order to determine its volume.

For example, to determine the volume of the composite solid shown, you can decompose the solid into two rectangular prisms and calculate the volume of each.

Volume of larger prism = \( 1.9 \times 2.8 \times 2.7 = 14.364 \text{ m}^3 \)

Volume of smaller prism = \( 1.3 \times 2.8 \times 0.5 = 1.82 \text{ m}^3 \)
To calculate the sum or difference of decimals, line up the decimals so that like place values are in the same column. Use the decimal point to help you correctly align.

A trailing zero was added to 1.82. Trailing zeros are a sequence of 0s in a decimal representation of a number, after which no non-zero digits follow. Trailing zeros do not affect the value of a number.

The volume of the composite solid is 16.184 cubic meters.

A net is a two-dimensional representation of a three-dimensional geometric figure. A net has all of these properties:

- The net is cut out as a single piece.
- All of the faces of the geometric solid are represented in the net.
- The faces of the geometric solid are drawn so that they share common edges.

The surface area of a polyhedron is the total area of all its two-dimensional faces.

For example, you can use the net to calculate the surface area of the right rectangular prism.

Determine the area of each unique face.

\[
4.3 \, \text{cm} \times 5.7 \, \text{cm} = 24.51 \, \text{cm}^2 \\
2.9 \, \text{cm} \times 5.7 \, \text{cm} = 16.53 \, \text{cm}^2 \\
4.3 \, \text{cm} \times 2.9 \, \text{cm} = 12.47 \, \text{cm}^2
\]

Determine the sum of all faces of the right rectangular prism.

\[
2(24.51) + 2(16.53) + 2(12.47) \\
= 49.02 + 33.06 + 24.94 \\
= 107.02
\]

The surface area of the right rectangular prism is 107.02 cm².
A slant height of a pyramid is the distance measured along a triangular face from the vertex of the pyramid to the midpoint, or center, of the base.

### Lesson 4

**Dividend in the House**

You can use a hundredths grid to model dividing decimals, such as $3.57 \div 3$.

First, shade hundredths grids to represent $3.57$.

Next, divide the shaded model into 3 equal groups.

One whole grid and 19 small squares are in each group. So, $3.57 \div 3 = 1.19$. 
You can also use a standard algorithm to divide 3.57 ÷ 3.

5 tenths divided into 3 equal groups is 1 tenth in each group with 2 tenths left over.

3 ones divided into 3 equal groups is 1 one in each group with 0 ones left over.

2 tenths and 7 hundredths is 27 hundredths. 27 hundredths divided into 3 equal groups is 9 hundredths in each group with 0 hundredths left over.

If you multiply or divide both the dividend and divisor by the same number, the quotient remains the same.

\[ 7.7 \div 3.5 = 77 \div 35 \]
\[ \frac{7.7}{3.5} = \frac{77}{35} \]

You can use what you know about dividing with decimals to solve problems about volume and surface area. For example, suppose the surface area of a cube is 48.24 square inches. Calculate the area of each face of the cube.

\[ \frac{8.04}{6} \]
\[ 48.24 \]

Since a cube has six congruent square faces, each face has an area of 8.04 square inches.
The lessons in this module build on your experiences solving addition and multiplication word problems and representing real-world situations on a coordinate plane. In this module, you will consider different ways quantities can be related to each other. You will learn about ratios and proportional relationships and reason about these relationships using various models, such as double number lines, ratio tables, and graphs. You will learn about percents, unit rates, and conversion rates.

**Topic 1  Ratios** ................................................................. M2-3
**Topic 2  Percents** ............................................................ M2-105
**Topic 3  Unit Rates and Conversions** ............................. M2-161
TOPIC 1
Ratios

Artists mix paints in specific ratios to produce different colors. Graphic designers and web developers use these mixtures, too. They can specify a color with an RGB value: a specific mix of red, green, and blue.

Lesson 1
It's All Relative
Introduction to Ratio and Ratio Reasoning ........................................... M2-7

Lesson 2
Going Strong
Comparing Ratios to Solve Problems .................................................. M2-25

Lesson 3
Oh, Yes, I Am the Muffin Man
Determining Equivalent Ratios.............................................................. M2-37

Lesson 4
A Trip to the Moon
Using Tables to Represent Equivalent Ratios......................................... M2-57

Lesson 5
They’re Growing!
Graphs of Ratios. .................................................................................. M2-69

Lesson 6
One Is Not Enough
Using and Comparing Ratio Representations ....................................... M2-85
TOPIC 1: RATIOS
Students begin this topic by learning about ratios as multiplicative comparisons, contrasting them with additive comparisons. "More than" and "less than" are examples of additive comparisons, whereas "twice as many" and "one half as many" are examples of multiplicative comparisons. Students learn about quantitative relationships represented by ratios and the different ways to represent ratios. They are introduced to percent as a special ratio, namely an amount per 100. Students use their initial understandings of ratio to model and determine equivalent ratios. To generate and display equivalent ratios in real-world and mathematical problems, they use tape diagrams, double number lines, scaling up and down, tables, and graphs.

Where have we been?
Students enter grade 6 with experience contrasting additive and multiplicative patterns and relationships. In prior grades, they wrote number sentences to represent multiplicative and additive scenarios. Students’ knowledge of equivalent fractions from elementary school provides the foundation for their developing understanding of equivalent ratios.

Where are we going?
This topic provides the basis for future learning of proportional relationships and slope. Students also graph equivalent ratios on the coordinate plane, a prerequisite for the more in-depth study of proportional relationships and direct variation in grade 7.

Using Double Number Lines to Determine Equivalent Ratios
A double number line shows two connected number lines. The number lines are connected by equivalent ratios. For example, this double number line shows that 3 corn muffins for $2.50 is equivalent to 6 corn muffins for $5.00.
Myth: There is one right way to do math problems.

Employing multiple strategies to arrive at a single, correct solution is important in life. Suppose you are driving in a crowded downtown area. If one road is backed up, then you can always take a different route. If you know only one route, then you’re out of luck.

Learning mathematics is no different. There may only be one right answer, but there are often multiple strategies to arrive at that solution. Everyone should get in the habit of saying: Well, that’s one way to do it. Is there another way? What are the pros and cons? That way, you avoid falling into the trap of thinking there is only one right way, because that strategy might not always work or there might be a more efficient strategy.

Teaching students multiple strategies is important. This helps students understand the benefits of the more efficient method. In addition, everyone has different experiences and preferences. What works for you might not work for someone else.

#mathmythbusted

Talking Points
You can further support your student’s learning by asking them to take a step back and think about a different strategy when they are stuck.

Questions to Ask
• What strategy are you using?
• What is another way to solve the problem?
• Can you draw a model?
• Can you come back to this problem after doing some other problems?

Key Terms
ratio
A ratio is a comparison of two quantities by division.

percent
A percent is a ratio whose denominator is 100. Percent is another name for hundredths.

rate
A rate is a ratio that compares two quantities that are measured in different units.

proportion
A proportion is an equation that states that two ratios are equal.
WARM UP
Write a fraction to represent each situation.
1. the number of boys in your math class compared to the number of students in the class
2. the number of girls in your math class compared to the number of students in the class
3. the number of students in your math class that are absent today compared to the total number of students in the class
4. the number of students in your math class that are in attendance today compared to the total number of students in your class

LEARNING GOALS
• Distinguish between additive and multiplicative relationships between two quantities.
• Understand the concept of a ratio: a ratio represents a multiplicative comparison between two quantities.
• Write ratios in different forms and use ratio language to represent relationships between two quantities.
• Distinguish between part-to-part and part-to-whole ratios.
• Understand that fractions are part-to-whole ratios between two quantities.
• Understand that percents are part-to-whole ratios between a quantity and 100.

KEY TERMS
• additive reasoning
• multiplicative reasoning
• ratio
• percent

In elementary school, you made many comparisons using addition and subtraction. You answered questions like, “If Johnny has 9 apples and Suzie has 12 apples, who has more apples?” Is there another way to compare values?
Predict the Score

The Crusaders and the Blue Jays just finished the first half of their basketball game.

<table>
<thead>
<tr>
<th></th>
<th>Halftime Score</th>
<th>Final Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crusaders</td>
<td>30</td>
<td>?</td>
</tr>
<tr>
<td>Blue Jays</td>
<td>20</td>
<td>?</td>
</tr>
</tbody>
</table>

1. Predict the final score. Explain your reasoning.
Robena and Eryn each predicted the final score of a basketball game between the Crusaders and the Blue Jays.

1. Analyze each prediction.

   a. Describe the reasoning that Robena and Eryn used to make each statement.

   **Robena**
   
<table>
<thead>
<tr>
<th></th>
<th>Halftime Score</th>
<th>Final Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crusaders</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>Blue Jays</td>
<td>20</td>
<td>40</td>
</tr>
</tbody>
</table>

   I think the final score will be double the score at halftime.

   **Eryn**
   
<table>
<thead>
<tr>
<th></th>
<th>Halftime Score</th>
<th>Final Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crusaders</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>Blue Jays</td>
<td>20</td>
<td>40</td>
</tr>
</tbody>
</table>

   I think the Crusaders will play hard enough to stay 10 points ahead of the Blue Jays.

   b. Which team had a better second half in each prediction?
One of the students used additive reasoning to make her comparison and the other used multiplicative reasoning. **Additive reasoning** focuses on the use of addition and subtraction for comparisons. **Multiplicative reasoning** focuses on the use of multiplication and division.

c. Which student used additive reasoning and which used multiplicative reasoning?

Vicki and her nephew Benjamin share the same birthday. They were both born on March 4.

Vicki: “Today I’m 40 years old, and you’re 10. I’m 4 times as old as you are!”

Benjamin: “Wow, you’re old!”

Vicki: “Yeah, but in 5 years, I’ll be 45, and you’ll be 15. Then I will only be three times as old as you.”

Benjamin: “I’m catching up to you!”

Vicki: “And 15 years after that, I’ll be 60 and you’ll be 30. Then I’ll only be twice as old as you!”

Benjamin: “In enough time, I’ll be older than you, Aunt Vicki!”

2. Is Vicki correct about how their ages change? Is Benjamin correct in thinking that he will eventually be older than his aunt?
3. The table represents the different statements from this problem situation. Let $V$ represent Vicki’s age and $B$ represent Benjamin’s age.

a. Complete the last column by identifying each relationship as either additive or multiplicative.

<table>
<thead>
<tr>
<th>Verbal</th>
<th>Numeric</th>
<th>Relationship</th>
</tr>
</thead>
<tbody>
<tr>
<td>Today I’m 40 years old, and you’re 10.</td>
<td>$V = 40$, $B = 10$</td>
<td>$V = B + 30$</td>
</tr>
<tr>
<td>I’m 4 times as old as you are!</td>
<td>$V = 40$, $B = 10$</td>
<td>$V = 4B$</td>
</tr>
<tr>
<td>Yeah, but in 5 years, I’ll be 45, and you’ll be 15.</td>
<td>$V = 45$, $B = 15$</td>
<td>$V = B + 30$</td>
</tr>
<tr>
<td>Then I will only be three times as old as you.</td>
<td>$V = 45$, $B = 15$</td>
<td>$V = 3B$</td>
</tr>
<tr>
<td>And 15 years after that, I’ll be 60 and you’ll be 30.</td>
<td>$V = 60$, $B = 30$</td>
<td>$V = B + 30$</td>
</tr>
<tr>
<td>Then I’ll only be twice as old as you!</td>
<td>$V = 60$, $B = 30$</td>
<td>$V = 2B$</td>
</tr>
</tbody>
</table>

b. At any point in this age scenario, which relationship does not change?
The school colors at Riverview Middle School are a shade of bluish green and white. The art teacher, Mr. Raith, knows to get the correct color of bluish green it takes 3 parts blue paint to every 2 parts yellow paint.

There are different ways to think about this relationship and make comparisons. One way is to draw a picture or model.

From the model, you can make comparisons of the different quantities.

- blue parts to yellow parts
- yellow parts to blue parts
- blue parts to total parts
- yellow parts to total parts

Each comparison is called a ratio. A ratio is a comparison of two quantities that uses division. The first two comparisons are part-to-part ratios because you are comparing the individual quantities. The last two comparisons are part-to-whole ratios because you are comparing one of the parts (either blue or yellow) to the total number of parts.

Suppose Mr. Raith needs 2 parts blue paint and 5 parts yellow paint to make green paint.

1. Compare the quantities of blue and yellow paint in Mr. Raith’s mixture by writing all possible ratios for each type.

   a. part-to-part ratios
   b. part-to-whole ratios

What is the difference between the part-to-part ratios that you wrote? What is the difference between the part-to-whole ratios that you wrote?
Ratios can be found all around you, even in your classroom! Just consider two different quantities.

For example, how many students in your class are wearing sneakers? How many students in your class are wearing another type of shoe?

1. Use a ratio to describe the relationship given.
   a. Write a part-to-part ratio comparing the number of students wearing sneakers to the number of students wearing a different type of shoe.

   b. Write a part-to-part ratio comparing the number of students wearing a shoe other than sneakers to the number of students wearing sneakers.

   c. Write a part-to-whole ratio comparing the number of students wearing sneakers to the total number of students in the class.

   d. Write a part-to-whole ratio comparing the number of students wearing a type of shoe other than sneakers to the total number of students in the class.
2. Search around your classroom for at least 3 pairs of quantities to compare. For each pair:

- Identify the two quantities that are being compared using ratios.
- Write all possible part-to-part and/or part-to-whole comparisons of the quantities.
- Identify each ratio as part-to-part or as part-to-whole.
- Be prepared to share your treasures from the Ratio Hunt with the class.

a. Quantities being compared:
   Ratio(s):

b. Quantities being compared:
   Ratio(s):

c. Quantities being compared:
   Ratio(s):
The Lanterton Middle School is adopting a new nickname. They have narrowed their search to the following two names: Tigers or Lions. To choose a nickname, they conducted a school-wide survey and tallied all the votes.

Each homeroom analyzed the results of the school-wide survey and reported the results in a different way.

**Homeroom 6A**
The votes for Tigers outnumbered the votes for Lions by a ratio of 240 to 160.

**Homeroom 6B**
There were 80 more votes for Tigers than Lions.

**Homeroom 7A**
The votes for Tigers outnumbered votes for Lions by a ratio of 3 to 2.

**Homeroom 7B**
3 out of 5 votes were for Tigers.

1. Describe the meaning of each statement. Then identify which describe ratios, and if so, whether the ratios are part-to-part or part-to-whole ratios.
Fractional form simply means writing the relationship in the form \( \frac{a}{b} \). Just because a ratio looks like a fraction does not mean it is representing a part-to-whole comparison.

**WORKED EXAMPLE**

Let’s consider the results reported by Homeroom 7A: “The votes for Tigers outnumbered votes for Lions by a ratio of 3 to 2.”

This comparison is an example of a part-to-part ratio expressed in words. There are two other ways you can express this part-to-part ratio.

**With a Colon**

3 votes for Tigers : 2 votes for Lions

**In Fractional Form**

\[
\frac{3 \text{ votes for Tigers}}{2 \text{ votes for Lions}}
\]

Next, let’s consider the results of the student vote as reported by Homeroom 7B: “3 out of 5 votes were for Tigers.”

2. Complete the part-to-whole and part-to-part ratios written in words. Then write each ratio with a colon and in fractional form. Label all quantities.

**Part-to-Whole Ratio**

<table>
<thead>
<tr>
<th>In Words</th>
<th>With a Colon</th>
<th>In Fractional Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 out of 5 votes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>were for Tigers.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>___ out of 5 votes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>were for Lions.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Part-to-Part Ratio**

<table>
<thead>
<tr>
<th>In Words</th>
<th>With a Colon</th>
<th>In Fractional Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>___ votes for Tigers for every</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 votes for Lions.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 votes for Lions for every ___</td>
<td></td>
<td></td>
</tr>
<tr>
<td>votes for Tigers.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Finally, let’s consider the results of the survey as reported by Homeroom 6A: “The votes for Tigers outnumbered the votes for Lions by a ratio of 240 to 160.

3. Complete the part-to-whole and part-to-part ratios written in words. Then write each ratio with a colon and in fractional form. Label all quantities.

<table>
<thead>
<tr>
<th>Part-to-Whole Ratio</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>In Words</td>
<td>With a Colon</td>
<td>In Fractional Form</td>
</tr>
<tr>
<td>_____ votes out of _____ votes were for Tigers.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>_____ votes out of _____ votes were for Lions.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Part-to-Part Ratio</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>In Words</td>
<td>With a Colon</td>
<td>In Fractional Form</td>
</tr>
<tr>
<td>_____ votes for Tigers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>_____ votes for Lions.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>_____ votes for Lions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>_____ votes for Tigers.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Based on the survey, which mascot name was preferred?
Consider each statement.

- There is an 80 percent chance of rain tomorrow.
- He ate $\frac{2}{5}$ of the cake.
- Sales tax in Greenmont is 7 percent.
- Three-fourths of the class is absent.

The situations described are examples of special types of ratios: fractions and percents.

Notice that when you write a ratio using the total number of parts, you are also writing a fraction. A fraction can be used as a ratio that shows a part-to-whole relationship.

A percent is a part-to-whole ratio where the whole is equal to 100. Percent is another name for hundredths. The percent symbol “%” means “per 100,” or “out of 100.” Therefore:

- 35% means 35 out of 100.
- 35% as a fraction is $\frac{35}{100}$.
- 35% as a decimal is 0.35.
- 35% as a ratio is 35 to 100, or 35 : 100.

You can shade 35 of the 100 squares on the hundredths grid to represent 35%.
1. Each hundredths grid represents a whole. Write a fraction and a percent to represent the shaded part of each grid.

a. 

b. 

c. 

d. 

e. 

f.
TALK the TALK

Writing and Classifying Ratios

There are several ways to compare two quantities and write ratios.

Ratios

\[
\text{With a Colon} \quad \frac{\text{part}}{\text{part}} : \frac{\text{part}}{\text{whole}}
\]

\[
\text{In Fractional Form} \quad \frac{\text{part}}{\text{part}} : \frac{\text{whole}}{\text{whole}} \quad \text{Fraction}
\]

1. Consider the statement: There are \( s \) sixth grade band members and \( t \) total sixth graders.

   a. Write a part-to-whole ratio using colon notation.

   b. Write a part-to-part ratio using colon notation.
2. A survey of sixth graders with pets revealed that $c$ students prefer cats and $d$ students prefer dogs.
   
a. How would you compare these two statements using part-to-part ratios?
   
b. How would you compare these two statements using part-to-whole ratios?
   
   
a. There are 9 girls for every 2 boys in art class.
   
b. Three out of every five students in art class will help paint the mural in the library.
c. There are 3 blueberry muffins to every bran muffin in a variety pack.

d. Of the 30 students in chorus, 14 of them play the piano.

e. The students planted 22 yellow daffodils and 10 white daffodils.
Assignment

Write
Describe two ratios in the real world. Write about at least one part-to-whole ratio and one part-to-part ratio.

Remember
A ratio is a comparison of two quantities using division.
A part-to-whole ratio compares a part of a whole to the total number of parts.
A part-to-part ratio compares parts.
A part-to-whole ratio is a fraction.
A percent is a fraction in which the denominator is 100.

Practice
The Lewis brothers just joined MovieQ, a club that provides them with free movies based on a list that they pre-select. The boys work together to pick the first 10 movies for their list, each brother adding to the list based on their favorite type of movie. John David puts 5 sports movies on the list; Parker chooses 3 war movies; and Stephen adds 2 comedies.

Write the ratio in colon and in fractional form to express each relationship.
1. sports movies to war movies
2. comedies to total movies
3. war movies to comedies
4. sports movies to total movies
5. comedies to sports movies
6. war movies to total movies

Stretch
During the 2015 regular season, the Pittsburgh Pirates won 98 baseball games, 53 of which were won in their home stadium. The regular season includes 162 games.

Write a ratio for each and identify it as part-to-whole or part-to-part.
1. number of games won to number of games lost
2. number of games won to number of games played
3. number of games lost to number of games played
4. number of games won at home to number of games won away
5. number of games won at home to number of games won
Review

1. A right rectangular prism is shown.

[Diagram of a right rectangular prism with dimensions 2/3 cm, 7/8 cm, and 1 cm]

a. Determine the volume of the prism.

b. Determine the surface area of the prism.

2. Estimate each sum or difference to the nearest whole number. Then calculate each sum or difference.

a. Cristina wants to purchase four items at the sporting goods store. The items she wants to buy are soccer cleats for $24.99, shin guards for $12.99, soccer socks for $4.49, and a soccer ball for $19.95. How much will the four items cost?

b. Jada and Tonya ran a 400-meter race. Jada ran the race in 75.2 seconds. Tonya ran the race in 69.07 seconds. How much faster did Tonya run the race?

3. Determine each product.

a. \(\frac{3}{8} \times \frac{4}{5}\)

b. \(2\frac{9}{10} \times \frac{2}{5}\)
LESSON 2: Going Strong   •   M2-25

Comparing Ratios to Solve Problems

WARM UP

Use reasoning to compare each pair of fractions.

1. \( \frac{6}{7} \) and \( \frac{8}{9} \)
2. \( \frac{7}{13} \) and \( \frac{5}{11} \)
3. \( \frac{4}{5} \) and \( \frac{4}{3} \)

You know how to write a ratio as a comparison of two quantities. How can you compare two ratios to make decisions in real-world situations?

LEARNING GOALS

• Apply qualitative ratio reasoning to compare ratios in real-world and mathematical problems.
• Apply quantitative ratio reasoning to compare ratios in real-world and mathematical problems.
• Compare and order part-to-part and part-to-whole ratios represented verbally, pictorially, and numerically.
Lemony-er Lemonade

Tammy’s glass of lemonade has a weaker tasting lemon flavor than Jen’s glass of lemonade. The shaded portion in each glass represents an amount of lemonade.

1. If one teaspoon of lemon mix is added to both Jen’s and Tammy’s glasses, which glass will contain the lemonade with the stronger lemon flavor? Explain your reasoning.
In this activity you will compare ratios without measuring or counting quantities. When you reason like this, it is called qualitative reasoning.

1. The shaded portion in each glass represents an amount of lemonade. Answer each question and explain your reasoning.

   a. Beth’s glass of lemonade has a weaker tasting lemon flavor than John’s glass of lemonade. If two ounces of water is added to Beth’s glass and one teaspoon of lemon mix is added to John’s glass, which glass will contain the lemonade with the stronger lemon flavor?

   b. Jimmy and Jake have glasses of lemonade that taste the same. If one teaspoon of lemon mix is added to each glass, which glass will contain the lemonade with the stronger lemon flavor?

   c. Jack’s glass of lemonade has a stronger tasting lemon flavor than Karen’s glass of lemonade. If one teaspoon of lemon mix is added to Karen’s glass and one ounce of water is added to Jack’s glass, which glass will contain the lemonade with the stronger lemon flavor?
2. Choose the correct statement to complete each sentence and explain your reasoning. If the answer cannot be determined, explain why not.

a. If Luke plans to use four more tablespoons of orange mix today than what he used yesterday to make the same amount of orange drink, his orange drink today would have:
   - a stronger tasting orange flavor.
   - a weaker tasting orange flavor.
   - a mix that has the same strength of orange taste as yesterday.

b. Dave and Sandy each made a pitcher of orange drink. Sandy’s pitcher is larger than Dave’s pitcher. Sandy used more orange mix than Dave. Dave’s orange drink has:
   - a stronger tasting orange flavor.
   - a weaker tasting orange flavor.
   - a mix that has the same strength of orange taste as Sandy’s drink.

c. If a race car travels more laps in less time than it did yesterday, its speed would be:
   - slower.
   - exactly the same.
   - faster.
The 6th grade students are making hot chocolate to sell at the Winter Carnival. Each homeroom suggested a different recipe.

<table>
<thead>
<tr>
<th></th>
<th>Recipe HR 6A</th>
<th>Recipe HR 6B</th>
<th>Recipe HR 6C</th>
<th>Recipe HR 6D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 cups milk</td>
<td>5 cups milk</td>
<td>3 cups milk</td>
<td>4 cups milk</td>
</tr>
<tr>
<td></td>
<td>3 T cocoa powder</td>
<td>8 T cocoa powder</td>
<td>4 T cocoa powder</td>
<td>7 T cocoa powder</td>
</tr>
</tbody>
</table>

1. Consider the given recipes to answer each question.

   a. Use reasoning to determine which recipe has the strongest chocolate taste and which recipe has the weakest chocolate taste.

   b. Show how you used ratio reasoning to order the recipes. Identify the ratios that you used as part-to-part or part-to-whole.

   c. Create a poster to explain your answer and strategies to the class. Prepare to share!
Suppose your class is in charge of providing punch at the upcoming open house. The Parent-Teacher Association bought lemon-lime soda and pineapple juice to combine for the punch, but they did not tell your class how much of each to use. Your classmates submitted suggestions for how to make the tastiest punch.

Cut out the punch ratio cards at the end of the lesson. Order the cards from the least lemon-lime concentration to the most lemon-lime concentration. If you think more than one card describes the same ratio of lemon-lime soda and pineapple juice, group those cards together.

1. Describe the strategies you used to sort and order the cards.
TALK the TALK

Put Me In, Coach

A soccer team has been awarded a penalty shot at the end of a tie game. If they make the penalty shot, they will win the league championship. The coach is considering three players to take the penalty. Amber has taken 4 penalty shots this season and has made 3 of them. Lindsay has taken 6 penalty shots and made 4. Li has taken 3 penalty shots and made 2.

1. Which player would you recommend take the penalty shot? Why?
**Punch Ratio Cards**

<table>
<thead>
<tr>
<th>A</th>
<th>For every lemon-lime soda, there is a pineapple juice.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>One-fourth of the punch is lemon-lime soda.</td>
</tr>
<tr>
<td>E</td>
<td>Half of the mixture is pineapple juice.</td>
</tr>
<tr>
<td>G</td>
<td>Lemon-lime soda : Pineapple juice = 4 : 5</td>
</tr>
<tr>
<td>I</td>
<td>For every lemon-lime soda, there are two pineapple juices.</td>
</tr>
<tr>
<td>K</td>
<td>Pineapple juice : lemon-lime soda = 3 : 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>B</th>
<th><img src="image1.png" alt="Image" /></th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td><img src="image2.png" alt="Image" /></td>
</tr>
<tr>
<td>F</td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>H</td>
<td><img src="image4.png" alt="Image" /></td>
</tr>
<tr>
<td>J</td>
<td>For every lemon-lime soda, there are 1(\frac{1}{2}) pineapple juices.</td>
</tr>
<tr>
<td>L</td>
<td>Three-fifths of the punch is pineapple juice.</td>
</tr>
</tbody>
</table>

**LESSON 2: Going Strong • M2-33**
Assignment

Write
Write two recipes for hot chocolate, each with a different ratio of chocolate mix to water or milk. Describe how the two recipes are similar and different.

Remember
One ratio can be less than, greater than, or equal to another ratio.

Practice
Megan is making fruit punch using fruit juice and ginger ale. She tries different combinations to get the mixture just right. If the ratio of fruit juice to ginger ale is too high, the punch is too fruity; if the ratio is too low, the punch is too gingery.

For each attempt, write a ratio Megan can try next time.
1. She tried 16 cups of fruit juice and 4 cups of ginger ale. That was too fruity.
2. She tried 10 cups of fruit juice and 8 cups of ginger ale. That was too gingery.
3. She tried 10 cups of fruit juice and 1 cup of ginger ale. That was too fruity.
4. She tried 8 cups of fruit juice and 4 cups of ginger ale. That was a little too gingery.
5. Based on Megan’s attempts in Questions 1-4, what might be a good ratio of fruit punch to ginger ale? Explain your thinking.

Stretch
Which of the given recipes will make cookies with the most chocolate chips per cookie? Order the recipes from the least chocolate chips per cookie to the most chocolate chips per cookies. Explain your answer.

Recipe 1: \( \frac{3}{4} \) cup of chips for a batch of 2 dozen cookies
Recipe 2: 1 cup of chips for a batch of 18 cookies
Recipe 3: \( \frac{3}{4} \) cup of chips for a batch of 12 cookies

LESSON 2: Going Strong • M2-35
Review

1. During the spring sports season, students at Hillbrook Middle School have the opportunity to either play baseball, run outdoor track, or play lacrosse. Of the 75 students at Hillbrook who play a spring sport, 30 run track, 25 play baseball, and 20 play lacrosse. Write the ratios and determine whether a part-to-part or part-to-whole relationship exists.
   a. track runners to baseball players
   b. track runners to total number of athletes

2. Determine the area of each face of a cube with the given surface area.
   a. 306.6 m²
   b. 450 in.²

3. Determine each sum.
   a. \( \frac{1}{6} + \frac{2}{3} \)
   b. \( \frac{5}{8} + \frac{1}{2} \)
WARM UP

Choose the correct statement to complete each sentence and explain your reasoning.

1. When the manager at Sweets-a-Plenty Bakery decides how many bakers are needed to bake muffins for a given day, she needs to consider the total number of muffins needed for the day.

   a. Making fewer muffins with more bakers will take:
      • less time.
      • an equal amount of time.
      • more time.

   b. Making more muffins in a shorter amount of time requires:
      • fewer workers.
      • an equal amount of workers.
      • more workers.

Informally comparing ratios, or qualitatively comparing ratios, is important. However, there are many instances when you need to make more specific comparisons. How can you use equivalent ratios in order to compare ratios more precisely?

LEARNING GOALS

- Use drawings to model and determine equivalent ratios.
- Reason about tape diagrams to model and determine equivalent ratios.
- Define and use rates and rate reasoning to solve ratio problems.
- Use scaling up and scaling down to determine equivalent ratios.
- Use double number lines to solve real-world problems involving ratios.

KEY TERMS

- equivalent ratios
- tape diagram
- rate
- proportion
- scaling up
- scaling down
- double number line

LESSON 3: Oh, Yes, I Am the Muffin Man  •  M2-37
Getting Started

Which Has More?

Consider the given representations to answer each question. Explain your reasoning.

1. Which dinner order has more pizza?

Order 1

Order 2

2. Which pattern has more stars?

Pattern 1

Pattern 2

3. Which pile of laundry has more shirts?

Pile 1

Pile 2

4. Which type of reasoning did you use for each question—additive or multiplicative? Explain why.
Kerri and her friends are going hiking. Kerri invites her friends to meet at her house for a quick breakfast before heading out on their hike. Kerri wants to offer muffins to her friends.

1. She knows that one muffin combo has four muffins that can feed four people.
   a. Draw a model showing the relationship between the muffin combo and the number of people it will feed.
   b. If Kerri invites 6 friends, how many muffin combos will she need? Draw a model to show how many muffin combo(s) she will need, and explain your answer.
   c. If Kerri has $2\frac{3}{4}$ muffin combos, how many friends can she feed? Draw a model to show how many friends she can feed, and explain your answer.
Let's consider a different variety pack.

In one muffin variety pack, two out of every five muffins are blueberry.

2. Draw a model to answer each question. Explain your reasoning.

a. How many muffins are blueberry muffins if there are a total of 25 muffins?

b. How many muffins are blueberry muffins if there are a total of 35 muffins?

c. How many total muffins are there if 8 muffins are blueberry?

As you solved these problems, you determined equivalent ratios. Equivalent ratios are ratios that represent the same part-to-part or part-to-whole relationship.
The local bakery sells muffins in variety packs of blueberry, pumpkin, and bran muffins. They always sell the muffins in the ratio of 3 blueberry muffins : 2 pumpkin muffins : 1 bran muffin.

1. Write the ratio that expresses each relationship. Identify each as a part-to-part or a part-to-whole ratio.

   a. blueberry muffins to total muffins
   b. pumpkin muffins to total muffins
   c. bran muffins to total muffins
   d. blueberry muffins to pumpkin muffins
   e. bran muffins to pumpkin muffins
   f. blueberry muffins to bran muffins

A ratio can be represented by drawing the objects themselves, but they also can be represented using a tape diagram. A tape diagram illustrates number relationships by using rectangles to represent ratio parts. A tape diagram representing the ratio of each type of muffin is shown.

2. What does each small rectangle represent in the given tape diagram?
Tape diagrams provide a visual representation of ratios, but they also can be used to solve problems.

**WORKED EXAMPLE**

Suppose you purchase an 18-pack of muffins. How many blueberry, pumpkin, and bran muffins will you purchase?

There are 6 muffins represented in the tape diagram, and you want 18 total muffins that are in the same ratio.

Therefore, to determine how many muffins you need to maintain the same ratio, you can divide 18 by 6.

\[ 18 \div 6 = 3 \]

Therefore, each rectangle will represent 3 muffins.

Blueberry: 3 3 3

Pumpkin: 3 3

Bran: 3

From the tape diagram, you can see that there are 9 blueberry muffins, 6 pumpkin muffins, and 3 bran muffins.

4. Suppose you purchase a 36-pack of muffins. Use the tape diagram to illustrate how many blueberry, pumpkin, and bran muffins you will receive.

Blueberry
Pumpkin
Bran

5. Suppose you wanted 20 pumpkin muffins in your variety pack. How many total muffins will be in your variety pack? Complete the tape diagram to determine the answer.

Blueberry
Pumpkin
Bran
6. The table shows the number of muffins in specific sized variety packs. Complete just the missing cells in the columns for the 6-pack and 36-pack of muffins.

<table>
<thead>
<tr>
<th>Total Number of Muffins</th>
<th>6</th>
<th>12</th>
<th>18</th>
<th>24</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Blueberry Muffins</td>
<td></td>
<td></td>
<td>9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of Pumpkin Muffins</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of Bran Muffins</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

7. Analyze the completed columns in the table.
   a. What do you notice about the numbers?
   b. How could you have determined the number of each type of muffin in the 18-pack without using the tape diagram?
   c. How could you have determined the number of each type of muffin in the 36-pack without using the tape diagram?
   d. Use what you noticed about the numbers in the table to complete the remaining columns for the number of each type of muffin in a 12-pack and in a 24-pack of muffins. Explain your strategy.
One of the rounds at the Math Quiz Bowl tournament is a speed round. A team of four students will represent Stewart Middle School in the speed round of the Math Quiz Bowl. One student of the team will be chosen to solve as many problems as possible in 20 minutes. The results from this week’s practice are recorded in the table.

<table>
<thead>
<tr>
<th>Student</th>
<th>Number of Correctly Solved Problems in a Specified Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kaye</td>
<td>4 problems correct in 5 minutes</td>
</tr>
<tr>
<td>Susan</td>
<td>7 problems correct in 10 minutes</td>
</tr>
<tr>
<td>Doug</td>
<td>1 problem correct in 2 minutes</td>
</tr>
<tr>
<td>Mako</td>
<td>3 problems correct in 4 minutes</td>
</tr>
</tbody>
</table>

1. Explain how Tia's reasoning and Lisa's reasoning about who should compete in the speed round are incorrect.

**Tia**

Susan should definitely compete in the speed round because she correctly solved the most problems.

**Lisa**

It took Susan the longest time to complete her problems. She should not compete in the speed round.
Each quantity in the table is a rate. A rate is a ratio that compares two quantities that are measured in different units. The rate for each student in this situation is the number of problems solved per amount of time.

**WORKED EXAMPLE**

Kaye’s rate is 4 problems correct per 5 minutes. This rate can be written as:

\[
\frac{4 \text{ problems correct}}{5 \text{ minutes}}.
\]

2. Write the rates for the other three team members.

   a. Susan
   b. Doug
   c. Mako

When two ratios or rates are equivalent to each other, you can write them as a proportion. A proportion is an equation that states that two ratios are equal. In a proportion, the quantities composing each part of the ratio have the same multiplicative relationship between them.

**WORKED EXAMPLE**

For example, you know that Kaye got four problems correct per 5 minutes. So, you can predict how many problems she could answer correctly in 20 minutes.

\[
\frac{4 \text{ problems correct}}{5 \text{ minutes}} \times 4 = \frac{16 \text{ problems correct}}{20 \text{ minutes}}.
\]

Kaye can probably answer 16 problems correctly in 20 minutes.
When you change one ratio to an equivalent ratio with larger numbers, you are scaling up the ratio. **Scaling up** means you multiply both parts of the ratio by the same factor greater than 1.

3. Use the definition of a ratio to verify that \( \frac{4}{5} \) is equivalent to \( \frac{16}{20} \).

Remember, one way to represent a ratio is in fractional form. It doesn’t matter which quantity is in the numerator or denominator; it matters that the unit of measure is consistent among the ratios.

**WORKED EXAMPLE**

You can write the proportion in a different way.

4. Determine the number of problems each student can probably solve in 20 minutes. Explain the scaling up you used to determine the equivalent ratio.

<table>
<thead>
<tr>
<th>Susan</th>
<th>Doug</th>
<th>Mako</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

5. Which team member is the fastest? Who would you pick to compete? Explain your reasoning.
The muffin variety packs baked by the Healthy for U Bakery come in a ratio of 2 blueberry muffins to 5 total muffins.

1. Scale up each muffin ratio to determine the unknown quantity.

   a. \[
   \frac{2 \text{ blueberry muffins}}{5 \text{ total muffins}} = \frac{20 \text{ blueberry muffins}}{? \text{ total muffins}}
   \]

   b. \[
   \frac{2 \text{ blueberry muffins}}{5 \text{ total muffins}} = \frac{30 \text{ blueberry muffins}}{? \text{ total muffins}}
   \]

   c. \[
   \frac{2 \text{ blueberry muffins}}{5 \text{ total muffins}} = \frac{? \text{ blueberry muffins}}{100 \text{ total muffins}}
   \]

   d. \[
   \frac{2 \text{ blueberry muffins}}{5 \text{ total muffins}} = \frac{50 \text{ blueberry muffins}}{? \text{ total muffins}}
   \]

   e. \[
   \frac{2 \text{ blueberry muffins}}{5 \text{ total muffins}} = \frac{? \text{ blueberry muffins}}{15 \text{ total muffins}}
   \]

   f. \[
   \frac{2 \text{ blueberry muffins}}{5 \text{ total muffins}} = \frac{28 \text{ blueberry muffins}}{? \text{ total muffins}}
   \]
When you change a ratio to an equivalent ratio with smaller numbers, you are *scaling down* the ratio. *Scaling down* means you divide both parts of the ratio by the same factor greater than 1, or multiply both parts of the ratio by same factor less than 1. Scaling down a ratio often makes it easier to understand.

2. Scale down each ratio to determine the unknown quantity.

   a. \[
   \frac{3 \text{ people}}{9 \text{ pizzas}} = \frac{?}{3 \text{ pizzas}}
   \]

   b. \[
   \frac{2 \text{ hoagies}}{6 \text{ people}} = \frac{1 \text{ hoagie}}{?}
   \]

   c. \[
   \frac{100 \text{ track shirts}}{25 \text{ people}} = \frac{?}{1 \text{ person}}
   \]

   d. \[
   \frac{60 \text{ tracks}}{5 \text{ CDs}} = \frac{?}{1 \text{ CD}}
   \]

   e. \[
   \frac{3 \text{ tickets}}{\$26.25} = \frac{1 \text{ ticket}}{?}
   \]

   f. \[
   \frac{12 \text{ hours}}{720 \text{ miles}} = \frac{4 \text{ hours}}{?}
   \]

   g. \[
   \frac{20 \text{ hours of work}}{\$240} = \frac{1 \text{ hour of work}}{?}
   \]

   h. \[
   \frac{3 \text{ gallons of red paint}}{2 \text{ gallons of yellow paint}} = \frac{?}{1 \text{ gallon of yellow paint}}
   \]
You know several strategies to determine the relationship between two quantities: drawing models, building tape diagrams, and scaling up or down. You can also use a double number line to visualize these relationships. A double number line is a model that is made up of two number lines used together to represent the ratio between two quantities. The intervals on each number line maintain the same ratio.

The Muffin Man Bakery offers two types of muffins—corn or cinnamon raisin. It costs the bakery $2.50 to make 3 corn muffins.

**WORKED EXAMPLE**

The ratio $2.50 : 3$ corn muffins is shown on the double number line.

You can see other equivalent ratios of cost : number of corn muffins by continuing to label each interval.

1. State the two new ratios of cost : number of corn muffins shown on the second double number line.
2. Describe the interval represented on each number line.

3. Use the double number line to determine equivalent ratios.
   a. Plot the new ratios. Explain your calculations.

   ![Number Line Diagram]

   b. What is the cost of making 12 corn muffins?

c. What is the cost of making 15 corn muffins?

d. What is the cost of making 18 corn muffins?

e. Describe any patterns you notice between the cost and the number of corn muffins made.
4. One pound of bananas costs $0.64. Use the double number lines to determine the cost for each quantity of bananas.

   \[
   \begin{array}{c|c}
   \text{bananas (lb)} & \text{cost ($)} \\
   \hline
   0 & 0 \\
   1 & 0.64 \\
   \end{array}
   \]

   a. \(2 \frac{1}{2}\) pounds
   
   b. \(\frac{1}{2}\) pound
   
   c. 2 pounds

5. The cost for The Muffin Man Bakery to make 4 cinnamon raisin muffins is $3.20. Use the double number line to determine equivalent ratios and answer each question. Explain your calculations.

   \[
   \begin{array}{c|c}
   \text{Cost ($)} & \text{Number of cinnamon raisin muffins} \\
   \hline
   0 & 0 \\
   \end{array}
   \]

   a. What is the cost to make 8 cinnamon raisin muffins?
   
   b. How many cinnamon raisin muffins are made for $12.80?
   
   c. What is the cost of making 12 cinnamon raisin muffins?
6. It takes 1 cup of sugar to make 12 oat bran muffins. Use the double number line to determine equivalent ratios and answer each question. Explain your calculations.

<table>
<thead>
<tr>
<th>Cups of sugar</th>
<th>Number of oat bran muffins</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>60</td>
</tr>
</tbody>
</table>

a. Plot the given ratio on the double number line.

b. How many oat bran muffins can be made using \( \frac{1}{2} \) cup of sugar? \( \frac{2}{3} \) cup of sugar? \( 1\frac{1}{2} \) cups of sugar?

c. How many cups of sugar are needed to make 3 muffins? 15 muffins? 9 muffins?
**TALK the TALK**

**Make a Choice**

Answer each question by using pictures, a tape diagram, or a double number line. Show all of your work and explain why you chose your strategy.

1. A T-shirt store keeps 7 white T-shirts on the shelves for every 3 purple T-shirts on the shelves.
   
a. How many white T-shirts are on the shelves if there are 15 purple T-shirts on the shelves?

b. How many purple T-shirts are on the shelves if there are 49 white T-shirts on the shelves?

c. How many white shirts are on the shelves if there are 40 total shirts (purple and white) on the shelves?

2. A grocery store advertises 4 pounds of apples for $6.00.
   
a. What is the cost for 3 pounds of apples?

b. What is the cost for 1 pound of apples?

c. How many pounds of apples can you purchase with $40.00?

Circle the question that your teacher has asked you to present to the class. Write at least 3 sentences to tell your classmates how you completed the work.
Assignment

Write
Compare and contrast tape diagrams and double number line models for representing ratio relationships. Use an example in your description.

Remember
Equivalent ratios are ratios that represent the same part-to-part or part-to-whole relationship.

A proportion is an equation that states that two ratios are equal. In a proportion, the quantities composing each part of the ratio have the same multiplicative relationship between them.

Scaling up means you multiply both parts of the ratio by the same factor greater than 1.

Scaling down means you divide both parts of the ratio by the same factor greater than one, or multiply both parts of the ratio by the same factor less than 1.

Practice
1. Ms. Yoto is putting together bags of fruit that contain 1 pear for every 2 apples. For each ratio given, create a picture module. Then, calculate the answer from your model, and explain your reasoning.
   a. How many apples are in the bag if there are a total of 9 pieces of fruit?
   b. How many apples are in the bag if there are a total of 15 pieces of fruit?
   c. How many pieces of fruit are there if there are 8 apples in the bag?
2. When creating playlists for dances, DJ Lew likes to maintain a ratio of 4 hip hop songs : 3 country songs : 1 slow song.
   a. Create a tape diagram to represent this ratio.
   b. Suppose DJ Lew has 40 songs on his playlist. Use the tape diagram to illustrate how many hip hop, country, and slow songs are on the playlist.
   c. Suppose DJ Lew wants to put 36 hip hop songs on the playlist. How many total songs will be on the playlist? Use a tape diagram to determine the answer.
3. Scale up or scale down each ratio to complete the proportion.
   a. \[
   \frac{2 \text{ teachers}}{26 \text{ students}} = \frac{8 \text{ teachers}}{?} \]
   b. \[
   \frac{12 \text{ inches}}{1 \text{ foot}} = \frac{?}{18 \text{ feet}} \]
   c. \[
   \frac{$39,000}{1 \text{ year}} = \frac{?}{3 \text{ years}} \]
   d. \[
   \frac{18 \text{ pencils}}{1 \text{ box}} = \frac{108 \text{ pencils}}{?} \]
   e. \[
   \frac{$40}{15 \text{ gallons}} = \frac{?}{3 \text{ gallons}} \]
   f. \[
   \frac{1200 \text{ boxes}}{9 \text{ truckloads}} = \frac{?}{3 \text{ truckloads}} \]
   g. \[
   \frac{280 \text{ beats}}{4 \text{ seconds}} = \frac{70 \text{ beats}}{?} \]
   h. \[
   \frac{520 \text{ cm}}{5.2 \text{ m}} = \frac{260 \text{ cm}}{?} \]
4. A mason is a person who builds structures with bricks, stone, cement block, or tile. A mason usually uses mortar to hold the bricks together. A general rule of thumb in masonry is that $2\frac{1}{2}$ bags of mortar are needed for every 100 cement blocks.
   a. Complete a double number line to determine the amount of mortar needed for each quantity of blocks.
   b. How many bags of mortar will a mason need for 350 blocks?
   c. How many bags of mortar will a mason need for 50 blocks?
   d. With $12\frac{1}{2}$ bags of mortar, how many blocks can the mason lay?

**Stretch**

Scale up or scale down each ratio to complete the proportion.

1. \[
\frac{7 \text{ cups of red dye}}{10 \text{ cups of yellow dye}} = \frac{?}{25 \text{ cups of yellow dye}}
\]

2. \[
\frac{?}{175 \text{ in.}} = \frac{542}{50 \text{ in.}}
\]

3. \[
\frac{47 \text{ feet}}{60 \text{ seconds}} = \frac{?}{45 \text{ seconds}}
\]

**Review**

1. In planning for the upcoming regional girls’ tennis tournament, Coach McCarter looked at her players’ statistics from the previous 2 months.
   - Sarah: 7 matches won, 3 matches lost
   - Sophie: 6 matches won, 4 matches lost
   - Grace: 7 matches won, 4 matches lost

   Based on their records, which player should Coach McCarter choose to attend the regional tournament? Explain your reasoning.

2. Hydrate sports drink calls for 7 scoops for every gallon of water. Sarah thinks the drink is too weak, and she wants to change it. Describe how she can change either the number of scoops or the amount of water to make the drink stronger.

3. Decide whether each amount is more closely related to volume or surface area.
   a. the amount of air in a room
   b. the amount of wood in a dog house.

4. Determine each product.
   a. \[\frac{2}{5} \times \frac{7}{3}\]
   b. \[4\frac{1}{6} \times 3\frac{4}{5}\]
A Trip to the Moon
Using Tables to Represent Equivalent Ratios

WARM UP
It takes 1 cup of milk to make a batch of 8 pancakes.

1. How many cups of milk does it take to make 16 pancakes?
2. How many cups of milk does it take to make 4 pancakes?
3. How many pancakes can be made with 4 cups of milk?

You have created equivalent ratios using pictures, tape diagrams, double number lines, and scaling up or scaling down. Are there other strategies you can use to determine equivalent ratios?

LEARNING GOALS
• Create and reason about tables of equivalent ratios.
• Use known values in a table to determine equivalent ratios.
• Solve problems by reasoning about graphs, diagrams, and tables of equivalent ratios.
I’m Your Density

Population density is a ratio that compares people to square miles. The graph shown gives the approximate population density of four U.S. states in 2015.

1. Which of the states shown has the greatest population density? Which state has the least population density? Explain what this means in your own words.

2. What is the population density of your state or your city? How does this compare with other states or cities?
Gravity is a natural force that attracts objects to each other. Gravity is the pull toward the center of an object like the Earth, a planet, or the Moon. Your weight on the Earth is the measure of the amount of gravitational attraction exerted on you by the Earth. The Moon has a weaker gravitational force than the Earth.

The ratio of weight on Earth : weight on the Moon is approximately $60 \text{ lb} : 10 \text{ lb}$.

You can use ratio tables to show how two quantities are related. Ratio tables are another way to organize information.

**WORKED EXAMPLE**

The table represents three equivalent ratios of weight on Earth (lb) : weight on the Moon (lb).

The ratio of 60 lb on Earth : 10 lb on the Moon is given.

<table>
<thead>
<tr>
<th>Weight on Earth (lb)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>30</td>
<td>90</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Weight on the Moon (lb)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>

Think about how the numbers in the table relate to each other.
1. Verify that adding the two existing equivalent ratios 60 lb on Earth : 10 lb on the Moon and 30 lb on Earth : 5 lb on the Moon produces the equivalent ratio 90 lb on Earth : 15 lb on the Moon by analyzing the quotient of each ratio. What do you notice?

2. Can you show a different strategy to determine the ratio of 90 lb on Earth : 15 lb on the Moon?

3. Howard, Carla, Mitsu, and Ralph each determined the weight of a 120-lb person on the Moon.
   a. Compare Howard’s and Carla’s strategies.

   **Howard**
   
   I can scale 60 up to 120 by multiplying by 2, so then I must also multiply 10 by 2 to get 20.

<table>
<thead>
<tr>
<th>Weight on Earth (lb)</th>
<th>60</th>
<th>30</th>
<th>90</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight on the Moon (lb)</td>
<td>10</td>
<td>5</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

   **Carla**
   
   I also got the ratio of 120 lb on Earth : 20 lb on the Moon.
b. Explain Mitsu’s reasoning. Then verify the ratio 120 lb on Earth : 20 lb on the Moon is a correct equivalent ratio.

Mitsu
I used the weights for a 30-lb person and a 90-lb person to obtain the weight of a 120-lb person.

<table>
<thead>
<tr>
<th>Weight on Earth (lb)</th>
<th>60</th>
<th>30</th>
<th>90</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight on the Moon (lb)</td>
<td>10</td>
<td>5</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

So that means 120 lb on Earth : 20 lb on the Moon.

c. Explain why Ralph’s reasoning is not correct.

Ralph
The difference between 90 and 120 is 30, so I just added 30 to 15 and got 45.

<table>
<thead>
<tr>
<th>Weight on Earth (lb)</th>
<th>90</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight on the Moon (lb)</td>
<td>15</td>
<td>45</td>
</tr>
</tbody>
</table>

I got the ratio of 120 lb on Earth : 45 lb on the Moon.
1. Explain Tracy’s strategy and determine the number of pizzas needed.

<table>
<thead>
<tr>
<th>Pizzas</th>
<th>2</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Students</td>
<td>9</td>
<td>45</td>
</tr>
</tbody>
</table>
2. Complete the table to show the number of pizzas to order given the number of students. Explain your calculations.

<table>
<thead>
<tr>
<th>Pizzas</th>
<th>2</th>
<th>10</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Students</td>
<td>9</td>
<td>45</td>
<td>450</td>
<td>135</td>
<td>270</td>
<td>225</td>
<td>900</td>
</tr>
</tbody>
</table>

3. Use your table of values to answer each question. Explain your calculations.

   a. How many students will 12 pizzas feed?

   b. How many students will 20 pizzas feed?

   c. How many students will 90 pizzas feed?
Parts and Wholes in Ratio Tables

Remember, the school colors at Riverview Middle School are a shade of bluish green and white. The art teacher, Mr. Raith, needs to mix different quantities of the green paint for several school projects. It takes 3 parts blue paint to 2 parts yellow paint to create the bluish green color. Carla needs 5 total pints of the bluish green paint, so she used 3 pints of blue paint and 2 pints of yellow paint.

Mr. Raith thought that the art students needed a table to help determine the correct amount of each color of paint for different projects—both large and small.

1. Complete the table with the correct amounts.
   Explain your reasoning.

<table>
<thead>
<tr>
<th>Amount of Bluish Green Paint Needed</th>
<th>5 pints</th>
<th>15 pints</th>
<th>18 pints</th>
<th>1.5 pints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yellow Paint</td>
<td>2 pints</td>
<td>8 pints</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Blue Paint</td>
<td>3 pints</td>
<td>12 pints</td>
<td>18 pints</td>
<td>1.5 pints</td>
</tr>
</tbody>
</table>

2. Examine Sally’s answer. Explain what is wrong with her thinking.

Sally

If I want 15 pints of bluish green paint, then I will need to add 10 to the original 5 total parts of bluish green to get 15. So, I should add 10 to each of the other numbers too to get 12 pints of yellow and 13 pints of blue.
Charlie said, “The table is helpful, but it cannot list every amount we might need for every painting project. I think if we multiply $\frac{2}{5}$ times the total amount of bluish green paint we need, we can determine the amount of yellow paint needed. If we multiply $\frac{3}{5}$ times the total amount of bluish green paint we need, we can determine the amount of blue paint needed.”


Charlene said, “I am thinking about this in a different way. The amount of blue paint is always $1\frac{1}{2}$ times as much as the amount of yellow paint.”

4. Is she correct in her thinking? Explain your reasoning.

Clifford said, “My thinking is related to Charlene’s. The yellow paint is $\frac{2}{3}$ of the blue paint.”

5. Is Clifford correct? Explain your reasoning.

6. How does Clifford’s thinking relate to Charlene’s thinking?
**Lollipop Recipe**

Consider the recipe for making one batch of lollipops.

- 2 cups granulated sugar
- \(\frac{2}{3}\) cup light corn syrup
- \(\frac{3}{4}\) cup water
- \(\frac{1}{4}\) teaspoon flavoring oil

1. The table represents the ratio of ingredients used to make lollipops. Complete the ratio table. Explain your calculations.

<table>
<thead>
<tr>
<th>Number of Batches</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sugar (c)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Corn syrup (c)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Water (c)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flavoring Oil (tsp)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. For each number of batches, describe how you can use addition to determine the amount of each ingredient needed.
   - a. 3 batches
   - b. 7 batches

3. For each number of batches, describe how you can use subtraction to determine the amount of each ingredient needed.
   - a. 3 batches
   - b. 7 batches
Assignment

Write
Describe how addition can be used with ratio tables to create equivalent ratios. Use examples in your explanation.

Remember
You can use a table to represent, organize, and determine equivalent ratios. You can use addition and multiplication to create equivalent ratios.

Practice
Each table represents the ratio of yellow daffodils to white daffodils for different garden displays. Complete each ratio table. Explain your calculations.

1. | Yellow daffodils | 9 | 36 | 45 |
   | White daffodils  | 15|    | 90 |

2. | Yellow daffodils | 7 | 28 |
   | White daffodils  | 6 | 12 | 42 |

3. | Yellow daffodils | 32| 16 |
   | White daffodils  | 48| 6  |12 |

4. | Yellow daffodils | 5 | 1  | 9 |
   | White daffodils  | 3 | 30 |

5. | Yellow daffodils | 105| 84 | 21 |
   | White daffodils  | 20 | 60 |

6. | Yellow daffodils | 55 | 22 | 77 |
   | White daffodils  | 25 | 10 | 5 |
**Review**

1. In tennis, an ace is a legal serve that cannot be returned and is not even touched by the opponent’s racket. Cecelia has an excellent serve. Last week, Cecelia hit 7 aces in 2 matches.
   a. If she plays 6 matches in the regional tournament, how many aces should she expect? Show your work.
   b. If she plays 10 matches in the regional tournament, how many aces should she expect? Show your work.

2. The winning time for the middle school 4-person 100-meter relay was 62.59 seconds. Suppose that each runner ran exactly the same amount of time. What would the time be for each runner?

3. Spring Hill Park is on a rectangular piece of land that measures 0.75 mile by 1.25 miles. Draw and label a rectangle to represent the park. Then determine the area of the park.

4. Determine each product.
   a. \(25 \times 0.31\)
   b. \(7.05 \times 3.72\)
LESSON 5: They’re Growing!   •   M2-69

They’re Growing!
Graphs of Ratios

WARM UP
A tree grows at a constant rate of 3 feet per year.
1. Write a ratio to represent the amount of growth in feet : the number of months.
2. Create a double number line that describes the growth of the tree every 12 months over a 48-month period.

LEARNING GOALS
• Plot ratios and equivalent ratios on a coordinate plane.
• Read equivalent ratios from graphs.
• Use ratio reasoning to determine equivalent ratios from graphs.
• Recognize the graphical representation of equivalent ratios.

Key Term
• linear relationship

So far, you have used scaling up or scaling down, tables, tape diagrams, pictures, and double number lines to determine equivalent ratios. How can you plot pairs of values on a coordinate plane and determine equivalent ratios?
Growing Rectangles

Consider a rectangle with a short side of length 2 units and a long side of length 3 units.

- In the first table, add the indicated number of units to both the long and short sides of the original rectangle.
- In the second table, multiply each original side length by the given value.
- For each rectangle, determine the ratio of the long side length : short side length.

<table>
<thead>
<tr>
<th></th>
<th>Original</th>
<th>+2 units</th>
<th>+3 units</th>
<th>+4 units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long side</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Short side</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratio</td>
<td>3 : 2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Original</th>
<th>×2 units</th>
<th>×3 units</th>
<th>×4 units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long side</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Short side</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratio</td>
<td>3 : 2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
1. What do you notice about the ratios for rectangles formed by adding to the sides of the rectangle?

2. What do you notice about the ratios for rectangles formed by multiplying the sides of the rectangle by a given value?
Analyze the rectangles at the end of the lesson.

1. Cut out each rectangle and sort into at least two piles. Share your sorts and your criteria.

2. Determine the side lengths of each rectangle. Label each rectangle with the length of its short side and the length of its long side.

3. Ava grouped together Rectangles A, C, E, F, G, and J. What do you think was her reasoning?

4. Gabriel’s sort was similar to Ava’s but he included Rectangle A with Rectangles B, D, H, I, and K. What do you think was his reasoning?

5. Complete the table for Ava’s Group and Gabriel’s Group. Write the ratios in fractional form, comparing the length of the short side to the length of the long side. Compare the ratios in each table. What do you notice?
6. Stack each group of rectangles with the smallest rectangle on top so that their longer sides are horizontal and their lower left corners align. What do you notice?

a. Ava’s Group

b. Gabriel’s Group

7. Attach each set of stacked rectangles to the appropriate coordinate grid, with the lower left corner of the rectangles at the origin of the grid.
8. Label the coordinates of the upper right corner of each rectangle. What do you notice about the coordinates in relation to your ratio?

9. Draw a line through the labeled points on each graph. What do you notice about which ordered pairs each line passes through?

Just as equivalent ratios can be represented using tables and double number lines, they can also be represented on the coordinate plane. The ratio $\frac{y}{x}$ is plotted as the ordered pair $(x, y)$. When you connect the points that represent the equivalent ratios, you form a straight line that passes through the origin, such as with Ava’s Group. In contrast, non-equivalent ratios are those represented by points that do not create a straight line through the origin, like Gabriel’s Group.
Let’s investigate how you can use a graph to determine other equivalent ratios, and see how all the representations are connected.

Stephanie runs a website for a local sports team that gets 50 views every hour. The table shows the ratio time : website views.

<table>
<thead>
<tr>
<th>Website Views</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (hr)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

The double number line shown represents the same data.

You can also represent equivalent ratios on a coordinate plane.

1. Label the remaining ratios on the graph.
WORKED EXAMPLE

Consider the question: How many views will Stephanie’s website have in 6 hours?

You know 4 different equivalent ratios from the original graph. The graph shows how to use the two ratios 2 hr : 100 views and 4 hr : 200 views to determine the equivalent ratio 6 hr : 300 views.

Stephanie’s website will have 300 views in 6 hours.

2. Describe how to determine how many views Stephanie’s website will have in 7 hours given each representation.

a. using the graph

b. using the table

c. using the double number lines
One way to analyze the relationship between equivalent ratios displayed on a graph is to draw a line to connect the points. You can also extend the line to make predictions of other equivalent ratios. Sometimes, all of the points on the line make sense. Other times when you draw a line, not all the points on the line make sense.

3. Draw a line through all the points you plotted on your graph. Do all the points on the line you drew make sense in this problem situation? Why or why not?

4. How do all the representations—tables, double number lines and graphs—show equivalent ratios? How are they similar? Describe some of the advantages of each representation.
Augie burns 225 calories for every 30 minutes he rides his bike.

1. Complete the table to chart the number of calories burned for different amounts of time. Then plot the table of values on the graph.

<table>
<thead>
<tr>
<th>Calories Burned</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (min)</td>
<td>30</td>
<td>10</td>
</tr>
</tbody>
</table>

2. Use your graph to answer each question.

   a. How many minutes would Augie have to bike to burn 150 calories?

   b. How many calories can he burn if he bikes for 25 minutes?

3. How was the graph helpful? Were there any limitations when using the graph to determine values?
TALK the TALK

To Graph or Not to Graph

Go back and examine all the graphs in this lesson.

1. What is similar about all of the graphs?

2. What is different about all the graphs?

3. Describe how you can use a line to analyze equivalent ratios. What are the benefits and limitations of using a graph to display and interpret ratios?
4. Complete the graphic organizer to demonstrate your understanding of ratios.

- **DEFINITION**
- **CHARACTERISTICS**
- **EXAMPLE**
- **NON-EXAMPLE**
Assignment

Write
Compare the graph of a ratio relationship with the graph of a relationship that is not represented by a ratio. How are they similar and different? Use an example to explain.

Remember
Just as equivalent ratios can be represented using tables and double number lines, they can also be represented in the coordinate plane. The ratio $\frac{y}{x}$ is plotted as the ordered pair $(x, y)$.

When you connect the points that represent the equivalent ratios, you form a straight line that passes through the origin. In contrast, non-equivalent ratios are those represented by points that cannot be connected by a straight line through the origin.

Practice
Create a graph to represent the values shown in each ratio table.

1. Weight (pounds) | 1 | 2 | 4 | 5
   Cost (dollars)  | 3 | 6 | 12 | 15

2. Time (hours) | 1 | 3 | 5 | 7
   Distance (miles) | 25 | 75 | 125 | 175

3. Time (minutes) | 15 | 30 | 45 | 60
   Calories | 80 | 160 | 240 | 320

4. Time (seconds) | 1 | 10 | 15 | 20
   Data (Mb) | 10 | 100 | 150 | 200

5. Time (minutes) | 15 | 30 | 45 | 60
   Distance (miles) | 1.5 | 3 | 4.5 | 6

6. Time (minutes) | 1 | 5 | 6 | 10
   Height (feet) | 6 | 30 | 36 | 60

Stretch
Create a scenario that could be represented by the relationship on the given graph. Describe the quantities, label the axes, and identify at least 4 equivalent ratios.
Review

1. Ellen loves to make her own clothes. With 45 yards of cloth, she can make 5 dresses. Create a double number line to explain your reasoning for each question.
   a. If Ellen has 72 yards of cloth, how many dresses can she make?
   b. If Ellen is going to make a dress for herself, how many yards of cloth does she need?

2. A customer used a $10 bill to pay for a 39-cent candy bar. Simone returned 61 cents. What mistake did Simone make? Explain how she should correct her mistake.

3. A grocery store is selling ground beef for $1.89 per pound. How much does it cost to buy 2.5 pounds?

4. Use estimation to place the decimal point in the correct position in each quotient.
   a. \(2.1 \div 48.72 = 232\)  
   b. \(8 \div 204.8 = 256\)
LESSON 6: One Is Not Enough
Using and Comparing Ratio Representations

WARM UP

1. Use the double number line to create a ratio table.

   \[\begin{array}{c|c|c|c|c}
   x & 0 & 30 & 60 & 90 & 120 \\
   y & 0 & 50 & 100 & 150 & 200 \\
   \end{array}\]

2. Create a scenario that fits the data on the double number line and ratio table. What ratio is associated with your scenario?

LEARNING GOALS

- Use graphs to compare ratios.
- Read and interpret ratios from graphs, double number lines, and tables.
- Use ratio and rate reasoning and multiple ratio models to solve problems.
- Compare representations of additive and multiplicative relationships.

You have used a variety of tools to determine equivalent ratios. How can you compare the different representations as you solve ratio problems?
Getting Started

Just-Right Ratios

Yana’s dad is trying to make his own bread. But each time he tries, the bread is either too dry because it has too much flour or too runny because it has too much water.

<table>
<thead>
<tr>
<th>Flour (cups)</th>
<th>Water (cups)</th>
<th>Dry / Runny</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>4</td>
<td>dry</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>runny</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>dry</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>runny</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>runny</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>dry</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>dry</td>
</tr>
<tr>
<td>12</td>
<td>9</td>
<td>runny</td>
</tr>
<tr>
<td>15</td>
<td>8</td>
<td>dry</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>runny</td>
</tr>
</tbody>
</table>

1. Use Xs to graph each attempt that was too dry. Use Os to graph attempts that were too runny.

2. Estimate a ratio that is “just right” and graph the ratio. Explain your reasoning.

3. Compare your graph with your classmates’ graphs. Did you all create the same graphs?
Comparing Ratio Graphs

The adult ticket price for admission into the Rollerville Amusement Park is $15. The table and graph show the ratio number of adult tickets : cost.

<table>
<thead>
<tr>
<th>Adult Tickets</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($)</td>
<td>15</td>
<td>30</td>
<td>45</td>
<td>60</td>
</tr>
</tbody>
</table>

The Rollerville Amusement Park has different charges for students and pre-school age children. Student tickets are $10. Pre-school age children tickets are $5.

1. Complete each table.

<table>
<thead>
<tr>
<th>Student Tickets</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pre-School Tickets</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Plot each set of equivalent ratios on the graph. Use a △ for the student tickets : cost ratios and a □ for pre-school tickets : cost ratios.

3. Draw three separate lines through the points that represent each ratio. What do you notice?

4. Do all the points on the line you drew make sense in this problem situation? Why or why not?

5. How can you tell by looking at the three lines which cost to ticket ratio is the highest and the lowest?
Choosing a Strategy to Solve Ratio Problems

You know different ways to think about ratios. So, you can use different strategies to solve problems.

1. The graph shown represents the number of gallons of water used for the number of times a toilet is flushed.
   a. Write each point on the graph as the ratio of gallons of water used : number of flushes.
   b. What do you notice about each ratio?
   c. How many gallons of water would be used if the toilet was flushed 8 times? Explain the method you used.
   d. How many times would the toilet be flushed to use 18 gallons of water? Explain the method you used.
   e. Did you use the same method to answer each question? If not, why?
2. The graph shown represents the number of gallons of water used for the number of loads of laundry washed.

   a. Write each point on the graph as the ratio of gallons of water used : number of loads of laundry.

   b. What do you notice about each ratio?

   c. How many gallons of water would be used for 7 loads of laundry? Explain the method you used.

   d. How many loads of laundry can be done if 45 gallons of water are used? Explain the method you used.

   e. Did you use the same method to answer each question? If not, why?
Comparing Ratios with Double Number Lines

Showerheads come in various styles and allow different rates of water to flow. The ratio gallons of water : time is given for three different showerhead models.

The first showerhead uses 20 gallons of water for every 5 minutes.

A second showerhead model uses 25 gallons of water for every 10 minutes.

A third showerhead model uses 8 gallons of water for every 5 minutes.

1. Which of the three showerheads used the least amount of water per minute?

2. Explain your reasoning using double number lines.
Two different jogging situations are given on the next two pages, along with a diagram showing the current relationship between the joggers.

1. At the end of the lesson, there are diagrams, equations, graphs, and verbal statements that each match one of the situations. Cut them out and tape them in their appropriate location. Then explain why each representation describes that relationship between the two joggers.

   a. Choose the diagram that shows the relationship between the joggers after 5 minutes.

   b. Choose the equation that represents the relationship between the two joggers.

   c. Choose the graph that models the relationship between the two joggers.

   d. Choose the type of relationship that exists between the two joggers.

In a proportion, the quantities composing each part of the ratio have the same multiplicative relationship between them. A multiplicative relationship is also known as a proportional relationship.
Two joggers are running at the same speed.

Diagram of the current position of the two joggers.

| J1 | J2 |

Diagram of the two joggers after 5 minutes.

Explanation:

Equation

Explanation:

Graph

Explanation:

Verbal Statement

Explanation:
**Jogger 2 runs twice as fast as Jogger 1.**

<table>
<thead>
<tr>
<th>Diagram of the current position of the two joggers</th>
</tr>
</thead>
<tbody>
<tr>
<td>J1</td>
</tr>
</tbody>
</table>

**Diagram of the two joggers after 5 minutes.**

Explanation:

**Equation**

Explanation:

**Graph**

Explanation:

**Verbal Statement**

Explanation:
TALK the TALK

In Goes the Kitchen Sink

You are given the ratio 6 red marbles : 9 blue marbles. For each model in the graphic organizer, write two ratios equivalent to the given ratio: one with numbers larger than the given and one with numbers smaller than the given. Show how you can use each model to determine the equivalent ratios.
Cut Out for Activity 6.4

\[ J_2 = J_1 + 10 \]

\[ J_2 = 2J_1 \]

Ratio Relationship

Additive Relationship
Assignment

Write
Describe the advantages and disadvantages of using double number lines, tape diagrams, equations, tables, and graphs to write, represent, and compare ratios.

Remember
You can use a number of different models, like graphs, tables, double number lines, and tape diagrams to analyze ratios and ratio relationships and to solve problems.

Practice
1. Use a graph to answer each question.
   a. Serena is driving to the mountains for a summer camping trip. She is traveling at a constant rate of 45 miles per hour. The graph shows the ratio time:distance. How far has Serena traveled after 4 hours?

   b. Cisco is exercising. The graph shows the ratio calories burned:time for Cisco. How many calories did Cisco burn in 30 minutes?

2. A recipe calls for 2 eggs for every 5 cups of milk. How many eggs were used if 20 cups of milk were used? Draw a double number line to answer the question.

3. Alberto is in charge of making lunch at a summer camp. He knows that 3 tuna casseroles will serve 15 campers. How many tuna casseroles should Alberto make to serve 35 campers?

Casseroles | 1 | 3 | 
---|---|---
Campers | 15 | 30 | 35

LESSON 6: One Is Not Enough • M2-97
**Stretch**
Four recipes for lemon-lime punch are represented on the graph shown. Which recipe has the strongest taste of lemon-lime? Which recipe has the weakest taste of lemon-lime? Use the graph to explain your answer.

**Review**
1. Morgan and her friends are testing their typing skills. Morgan took an online typing test to compare her typing speed with her friends’ speeds. During the 2 minute test, she typed 144 words. Her friend, Elizabeth, took a longer test; she typed 150 words in 3 minutes. Their other friend, Ruth, typed 65 words in 1 minute.
   a. Create a ratio table to show each girl’s typing speed for 1 through 6 minutes.
   b. Plot each set of equivalent ratios on a coordinate plane. Use × to denote Morgan's typing speed, □ to denote Elizabeth’s typing speed, and ★ to denote Ruth’s typing speed.
   c. Draw three separate lines through the points that represent each ratio. What do you notice?
   d. Who is the fastest typist? Who is the slowest typist? Explain how you can tell by looking at the three lines on your graph.

2. Morgan uses her typing skills to write a research paper for her history class. When she hits “Print,” she realizes that her printer is broken—for every 5 pages she attempts to print, the printer messes up 3 of them! Create a ratio table to display the number of pages her printer would mess up. Then create a graph for your table of values. Be sure to label the axes and title the graph.

3. Determine the surface area of each figure based on the measurements of its net.
   a. ![Net Diagram](image)
   b. ![Net Diagram](image)

M2-98 • **TOPIC 1: Ratios**
Ratios Summary

**KEY TERMS**

- additive reasoning
- multiplicative reasoning
- ratio
- percent
- equivalent ratios
- tape diagram
- rate
- proportion
- scaling up
- scaling down
- double number line
- linear relationship

---

**Additive reasoning** focuses on the use of addition and subtraction for comparisons. **Multiplicative reasoning** focuses on the use of multiplication and division.

A **ratio** is a comparison of two quantities that uses division.

For example, an art teacher knows it takes 3 parts blue paint to every 2 parts yellow paint to create a certain shade of bluish green. This is represented by the model shown.

---

A part-to-whole ratio compares a part of a whole to the total number of parts. 3 to 5 is a part-to-whole ratio comparing blue parts to total parts.

**With a Colon**

3 blue parts : 5 total parts

**In Fractional Form**

\[
\frac{3 \text{ blue parts}}{5 \text{ total parts}}
\]

A part-to-part ratio compares individual quantities. 2 to 3 is a part-to-part ratio comparing yellow parts to blue parts.

**With a Colon**

2 yellow parts : 3 blue parts

**In Fractional Form**

\[
\frac{2 \text{ yellow parts}}{3 \text{ blue parts}}
\]
Fractional form simply means writing the relationship in the form $\frac{a}{b}$. Just because a ratio looks like a fraction does not mean it represents a part-to-whole comparison. Only a part-to-whole ratio is a fraction.

A **percent** is a part-to-whole ratio where the whole is equal to 100. The percent symbol “%” means “per 100,” or “out of 100.”

35% means 35 out of 100.
35% as a fraction is $\frac{35}{100}$.
35% as a decimal is 0.35.
35% as a ratio is 35 to 100, or 35 : 100.

One ratio can be less than, greater than, or equal to another ratio.

For example, the shaded portion in each glass represents an amount of lemonade. Suppose Jimmy and Jake have glasses of lemonade that taste the same. If one teaspoon of lemonade mix is added to each glass, Jake’s glass will now contain lemonade with a stronger lemon flavor. The ratio of lemon mix to lemonade is greater in Jake’s glass because he had less lemonade in the glass to begin with.

The cups represent different ratios of lemon-lime soda to pineapple juice in two different punches.

**Punch A**

**Punch B**

The concentration of lemon-lime soda in Punch A is $\frac{2}{6}$. The concentration of lemon-lime soda in Punch B is $\frac{2}{5}$. Punch B has a greater concentration of lemon-lime soda.
Equivalent ratios are ratios that represent the same part-to-part or part-to-whole relationship. You can use a tape diagram to help determine equivalent ratios. A tape diagram illustrates number relationships by using rectangles to represent ratio parts.

For example, the ratio of muffins in a variety pack is 3 blueberry muffins : 2 pumpkin muffins : 1 bran muffin and is represented by the tape diagram shown.

To determine how many of each type of muffin are in an 18-pack of muffins, you need to maintain the same ratio. Since there are 6 muffins represented in the tape diagram, divide 18 by 6.

Since $18 \div 6 = 3$, each rectangle will represent 3 muffins. The ratio of muffins in an 18-pack will be 9 blueberry muffins : 6 pumpkin muffins : 3 bran muffins.

A rate is a ratio that compares two quantities that are measured in different units.

For example, Kaye can answer 4 problems correctly in five minutes. This rate can be written as $\frac{4 \text{ problems correct}}{5 \text{ minutes}}$. When two ratios or rates are equivalent to each other, you can write them as a proportion. A proportion is an equation that states that two ratios are equal. In a proportion, the quantities composing each part of the ratio have the same multiplicative relationship between them.

You can predict how many problems Kaye could answer correctly in 20 minutes.

Kaye can probably answer 16 problems correctly in 20 minutes.
When you change one ratio to an equivalent ratio with larger numbers, you are scaling up. **Scaling up** means you multiply both parts of the ratio by the same factor greater than 1. When you change a ratio to an equivalent ratio with smaller numbers, you are scaling down. **Scaling down** means you divide both parts of the ratio by the same factor greater than one, or multiply both parts of the ratio by the same factor less than one.

A **double number line** is a model that is made up of two number lines used together to represent the ratio between two quantities. The intervals on each number line maintain the same ratio.

For example, the ratio $2.50 : 3$ corn muffins is shown on the double number line. You can see other equivalent ratios of *cost : number of corn muffins* by continuing to label each interval.

<table>
<thead>
<tr>
<th>Number of corn muffins</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost ($)</td>
<td>2.50</td>
<td>5.00</td>
<td>7.50</td>
<td></td>
</tr>
</tbody>
</table>

**LESSON 4**

**A Trip to the Moon**

You can use ratio tables to show how two quantities are related. Ratio tables are another way to organize information.

You can use a table to represent, organize, and determine equivalent ratios. You can use addition and multiplication to create equivalent ratios.

For example, the table shown represents three equivalent ratios of *weight on Earth (lb) : weight on the Moon (lb)*. The ratio of 60 lb on Earth : 10 lb on the Moon is given. One equivalent ratio was determined by dividing the original ratio by 2. Another was determined by adding two equivalent ratios.

<table>
<thead>
<tr>
<th>Weight on Earth (lb)</th>
<th>60</th>
<th>30</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight on the Moon (lb)</td>
<td>10</td>
<td>5</td>
<td>15</td>
</tr>
</tbody>
</table>

+2

add
Equivalent ratios can also be represented on the coordinate plane. The ratio $\frac{y}{x}$ is plotted as the ordered pair $(x, y)$. When you connect the points that represent equivalent ratios, you form a straight line that passes through the origin. In contrast, non-equivalent ratios are those represented by points that do not create by a straight line through the origin. When a set of points graphed on a coordinate plane forms a straight line, a linear relationship exists.

For example, the table charts the number of calories Valerie burns for different amounts of time.

<table>
<thead>
<tr>
<th>Calories Burned</th>
<th>240</th>
<th>80</th>
<th>480</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (mins)</td>
<td>30</td>
<td>10</td>
<td>60</td>
<td>50</td>
</tr>
</tbody>
</table>

The values are plotted on the graph.

The graph shows that Valerie would burn 200 calories after bicycling for 25 minutes and that it would take between 35 and 40 minutes of bicycling for her to burn 300 calories.
You can use a number of different models, like graphs, tables, double number lines, and tape diagrams to analyze ratios and ratio relationships and to solve problems.

For example, by comparing the graphed lines that represent each ratio of number of tickets: cost, you can tell that the cost to ticket rate is the greatest for adults because it has the steepest line. Likewise, the cost to ticket rate is the lowest for the pre-schoolers because it has the least steep line.
TOPIC 2

Percents

This image can represent the 75-day development of a carrot in 15-day increments. Each stage represents \( \frac{15}{75} \), or 20%, of the carrot’s growth.

Lesson 1
We Are Family
Percent, Fraction, and Decimal Equivalence ................................. M2-109

Lesson 2
Warming the Bench
Using Estimation and Benchmark Percents ................................. M2-123

Lesson 3
The Forest for the Trees
Determining the Part and the Whole in Percent Problems .............. M2-137
TOPIC 2: PERCENTS
In this topic, students transition from thinking about ratio relationships in general to focusing on a special ratio relationship: percent. Students learn that a percent can be defined multiple ways: as a ratio; as a decimal to the hundredths place; and as a part-to-whole relationship in which the whole is 100. Students use their knowledge of fractions and decimals and their intuitive understanding of percents to write and compare rational numbers written in these three different forms. They complete number lines of common fractions, decimals, percent equivalences, connecting to prior work with benchmark fractions and decimals. Throughout this topic, students continue to develop their fluency with whole numbers, fractions, decimals, area, and volume in the context of solving mathematical and real-world problems.

Where have we been?
Students have used the relationship between decimals and fractions to write decimals as fractions, and they have used benchmark fractions and decimals to understand ordering of numbers. This topic provides students with similar experiences using this new representation: percents. Because percent is a special ratio, students continue to use the strategies and reasoning developed in the prior topic to solve percent problems.

Where are we going?
Percents are very useful, not only in mathematics, but in everyday life and work. In grade 7, students will use the foundation they establish here to solve more advanced percent problems, including problems involving discounts, tax, interest, percent increase or decrease, and tips.

Using a Hundredths Grid to Represent a Percent
A hundredths grid is a 10 by 10 grid of squares, which are shaded to show different percents. Hundredths grids emphasize that percents are ratios of amounts to 100. When the entire grid is shaded, it represents 1 whole, or 100%.

45%
Myth: Students only use 10% of their brains.

Hollywood is in love with the idea that humans only use a small portion of their brains. This notion formed the basis of the movies Lucy (2014) and Limitless (2011). Both films ask the audience: Imagine what you could accomplish if you could use 100% of your brain!

Well, this isn’t Hollywood, and you’re stuck with an ordinary brain. The good news is that you do use 100% of your brain. As you look around the room, your visual cortex is busy assembling images; your motor cortex is busy moving your neck; and all of the associative areas recognize the objects that you see. Meanwhile, the corpus callosum, which is a thick band of neurons that connect the two hemispheres, ensures that all of this information is kept coordinated. Moreover, the brain does this automatically, which frees up space to ponder deep, abstract concepts like mathematics!

#mathmythbusted

Talking Points
A common error that students make when working with part-to-whole ratios (like percents and fractions) is to forget about the whole. Look for ways to remind your student about this common mistake.

For example, this model shows 24 shaded squares. Students might say that 24% is shaded.

But the whole is not 100, it’s 40. So, \( \frac{24}{40} \), or 60%, is shaded. Also, more than half is shaded, so it has to be more than 50%.

Key Term
benchmark percents
A benchmark percent is a percent that is commonly used, such as 1%, 5%, 10%, 25%, 50%, and 100%.
LEARNING GOALS
• Write equivalent fractions, decimals, and percents.
• Model percents as rates per 100 on a hundredths grid.
• Explain the similarities and differences among percents, fractions, and decimals.

WARM UP
Rewrite each fraction as an equivalent fraction with a denominator of 100.

1. $\frac{1}{10}$
2. $\frac{2}{5}$
3. $\frac{3}{20}$
4. $\frac{24}{40}$

You have learned that percents are special types of ratios. How are percents like another special type of ratio—fractions? You also know that fractions can be written as decimals. How are percents like decimals?
Getting Started

They’re All Part of the Same Family

Percents are everywhere! Write one or two sentences to explain the meaning of each statement.

1. Big Sale! 25% discount on all regularly priced items.

2. There is a 60 percent chance of snow tomorrow.

3. The star of the high school basketball team makes 80 percent of her free throws.

4. I scored an 80% on the 20-question test.
The sixth grade class is planning a field trip to Philadelphia. To decide which historical site they will visit, the 100 sixth-graders completed a survey.

1. The results of the survey are provided in the table. Complete the Ratio, Fraction, Decimal, and Grid columns with these representations of the survey results:
   - a ratio using colon notation
   - a fraction in lowest terms
   - a decimal
   - a shaded grid
   - an equivalent percent
### Which excursion would you like to take while in Philadelphia?

<table>
<thead>
<tr>
<th>Ratio</th>
<th>Fraction</th>
<th>Decimal</th>
<th>Grid</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>35 of the students chose the Liberty Bell.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22 of the students chose Independence Hall.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 of the students chose the National Constitution Center.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13 of the students chose the Betsy Ross House.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 of the students chose Reading Terminal Market.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Are you planning on going on the trip?

<table>
<thead>
<tr>
<th>Ratio</th>
<th>Fraction</th>
<th>Decimal</th>
<th>Grid</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 of the students responded Yes.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Recall that a percent can be a special part-to-whole ratio with a whole of 100. You can also think of a percent as a fraction in which the denominator is 100.

Percents, fractions, and decimals can be used interchangeably.

WORKED EXAMPLE

You can write 15 out of 100 as the fraction $\frac{15}{100}$ or $\frac{3}{20}$.

Written as a decimal, 15 out of 100 is 0.15.

Because percent means “out of 100,” 15 out of 100 can also be written as 15%.

2. Express each of the ratios in the survey as a percent in the last column of the table.

3. Write a summary of the results of the student survey using percents.

4. Look at the percents and the decimals you wrote for Question 1 to determine a pattern. Use this pattern to describe how you can write any percent as a decimal.
5. Write each percent as a decimal.
   a. 80%  
   b. 3%
   c. 12.5%  
   d. 125%

6. Write each decimal as a percent.
   a. 0.4  
   b. 0.07
   c. 0.7381  
   d. 1.52

When the denominator is a factor of 100, scale up the fraction to write it as a percent. When the denominator is not a factor of 100, you can divide the numerator by the denominator to write the fraction as a decimal, which you can then write as a percent.

7. Write each fraction as a percent. Round your answer to the nearest tenth of a percent.
   a. $\frac{4}{5}$  
   b. $\frac{3}{10}$
   c. $\frac{3}{8}$  
   d. $\frac{3}{2}$
8. Label each mark on the number line with a fraction, decimal, and percent. Make sure your fractions are in lowest terms.

a. 

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1/3</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>1</td>
<td>0.66</td>
<td>100%</td>
</tr>
</tbody>
</table>

b. 

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1/8</td>
<td>0.125</td>
<td>25%</td>
</tr>
<tr>
<td>0.625</td>
<td>1/4</td>
<td>25%</td>
</tr>
<tr>
<td>0.75</td>
<td>3/4</td>
<td>75%</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>100%</td>
</tr>
</tbody>
</table>

c. 

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1/10</td>
<td>0.1</td>
<td>10%</td>
</tr>
<tr>
<td>2/5</td>
<td>0.4</td>
<td>40%</td>
</tr>
<tr>
<td>7/10</td>
<td>0.7</td>
<td>70%</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>100%</td>
</tr>
</tbody>
</table>
On Saturday, Melanie won 3 out of 4 of her tennis matches at the Redstone Tournament. On Sunday, she won 1 out of 4 of her matches at the Mesa Tennis Tournament.

Each student summarized Melanie's record over the weekend.

**Patrick**

Melanie won 100% of her matches!

\[
\frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1
\]

1. What is wrong with Patrick's reasoning?

**Laura**

Melanie won 50% of her matches!

\[
\frac{\text{3 matches won}}{\text{4 matches played on Sat}} + \frac{\text{1 match won}}{\text{4 matches played on Sun}} = \frac{\text{4 matches won}}{\text{8 total matches played}}
\]

2. How did Laura make her reasoning explicit?

**Jonathon**

Melanie won 4 out of 8 matches played.

\[
\frac{\text{3 matches won}}{\text{4 matches played on Saturday}} : \frac{\text{1 match won}}{\text{4 matches played on Sunday}} = \frac{\text{4 matches won}}{\text{8 total matches played}}
\]
ACTIVITY 1.3

Matching Percents, Fractions, and Decimals

It’s time to play The Percentage Match Game. In this game, you will use your knowledge of percents, fractions, and decimals.

Rules of the Game:

• For this 2-person game, 1 person needs to cut out the cards located at the end of the lesson.

• Lay out all the cards facedown.

• The first player chooses any card. That player then turns over another card to see if it is an equivalent match. If the value on the two cards are equivalent, then the match is put into the player’s pile. The first player then picks again and repeats the process until a match is not found.

• If the first player does not have an equivalent match, turn the cards back over. It is the second player’s turn. The same process for picking and matching cards described is now followed by the second player.

• Continue taking turns until all possible matches are made.

• The player with the greater number of correct equivalent matches wins the game.

3. What is the same about Laura’s and Jonathon’s reasoning? What is different?

4. Why do Laura's and Jonathon answers make sense?
TALK the TALK

Family Resemblances

Percents, fractions, and decimals can be used interchangeably. The chart shows some common equivalent fractions, decimals, and percents.

| Common Equivalent Fractions, Decimals, and Percents |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Fraction        | 1/5             | 1/4             | 1/3             | 2/5             | 1/2             | 3/5             | 2/3             | 3/4             | 4/5             |
| Decimal         | 0.2             | 0.25            | 0.3             | 0.4             | 0.5             | 0.6             | 0.6             | 0.75            | 0.8             |
| Percent         | 20%             | 25%             | 33 1/3%         | 40%             | 50%             | 60%             | 66 2/3%         | 75%             | 80%             |

1. How are percents similar to decimals? How are percents and decimals different?

2. How are percents similar to fractions? How are percents and fractions different?

3. How are percents similar to ratios? How are percents and ratios different?
<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percentage</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{5}$</td>
<td>0.6</td>
<td>60%</td>
<td>33%</td>
</tr>
<tr>
<td>$\frac{3}{10}$</td>
<td>$\frac{1}{3}$</td>
<td>60%</td>
<td>33%</td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>$\frac{2}{6}$</td>
<td>12.5%</td>
<td>0.3</td>
</tr>
<tr>
<td>$\frac{1}{10}$</td>
<td>$\frac{1}{2}$</td>
<td>1%</td>
<td>50%</td>
</tr>
<tr>
<td>$\frac{1}{5}$</td>
<td>$\frac{3}{4}$</td>
<td>60%</td>
<td>66.6%</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.75</td>
<td>50%</td>
<td>75%</td>
</tr>
</tbody>
</table>
**Assignment**

**Write**
Define percent in your own words. Then describe how to write fractions and decimals as percents.

**Remember**
Percent can be used to represent a part-to-whole relationship with a whole of 100. The symbol % means “out of 100.”

**Practice**

1. Label each mark on the number line with a fraction, decimal, and percent. Make sure your fractions are in lowest terms.

   ![](image)

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>0%</td>
</tr>
<tr>
<td>1/5</td>
<td>0.2</td>
<td>20%</td>
</tr>
<tr>
<td>3/5</td>
<td>0.6</td>
<td>60%</td>
</tr>
<tr>
<td>1</td>
<td>1.0</td>
<td>100%</td>
</tr>
</tbody>
</table>

2. The table shows the portion of sixth graders at your school who have a particular number of siblings. Complete the table by representing each portion as a part-to-whole ratio, a fraction, a decimal, and a percent. Make sure your ratios and fractions are in lowest terms.

<table>
<thead>
<tr>
<th>Number of Siblings</th>
<th>Ratio</th>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>3/20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>20%</td>
</tr>
<tr>
<td>2</td>
<td>3:8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>0.24</td>
<td></td>
</tr>
<tr>
<td>4 or more</td>
<td></td>
<td>7/200</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Review**

1. Ellen loves to make her own clothes. With 45 yards of cloth, she can make 5 dresses. To accessorize her new dresses, Ellen decides to order textured stockings from an online store. The graph shows the costs of orders of stockings.

![Graph showing costs of stockings](image)

a. If Ellen has 18 yards of cloth, how many dresses can she make? Create a double number line to show your answer.

b. If Ellen wants to make dresses for 6 cousins, how many yards of cloth does she need? Create a double number line to show your answer.

c. Write each point on the graph as a ratio of number of pairs of stockings : total cost of the order.

d. How much would an order of 8 pairs of stockings cost? Explain the method you used.

2. Use the standard algorithm to determine each quotient.

   a. $885 \div 6$
   
   b. $9218 \div 330$

---

**Stretch**

Write each percent as a fraction and as a decimal. Explain your strategy.

1. 117%
2. 1048%
3. 0.15%
4. 0.0593%
LEARNING GOALS
• Order fractions, decimals, and percents.
• Estimate the percent of a quantity shaded in a model.
• Use benchmark percents to calculate common percents of quantities.
• Estimate percents using benchmarks.

KEY TERM
• benchmark percents

WARM UP
Compute each product.
1. \( \frac{1}{10} \times 350 \)
2. \( \frac{1}{100} \times 350 \)
3. \( \frac{1}{10} \times 670 \)
4. \( \frac{1}{100} \times 670 \)

You have used reasoning to calculate areas, volumes, decimal and fractional values, and equivalent ratios. How can reasoning be used to solve percent problems?
Putting It All in Perspective

In your opinion, what does each famous quotation or saying really mean?

1. “Genius is one percent inspiration and ninety-nine percent perspiration.”
   - Thomas Edison

2. “Success is 99 percent failure.”
   - Soichiro Honda

3. “You miss 100 percent of the shots you never take.”
   - Wayne Gretzky

4. "Always give 110%. It's the extra 10% that everyone remembers."
   - Frank Sonnenberg
Each student has been given a note card that contains a number expressed as a fraction, decimal, or percent.

As a class, order the set of numbers from least to greatest.

1. Explain the strategies used by your class to order the numbers.

Noah and Dylan were assigned the numbers 0.0\(\overline{6}\) and 0.1\% but they disagreed on which was larger. Noah says that 0.0\(\overline{6}\) is less than 0.1, so 0.0\(\overline{6}\) is less than 0.1\%.

Dylan says that since 0.1\% is the same as as 0.001 and 0.001 is less than 0.0\(\overline{6}\), 0.1\% is less than 0.0\(\overline{6}\).

2. Who is correct? Explain your reasoning.

3. Order the numbers from least to greatest.

\[0.99, \frac{1}{9}, 0.17, 95\%, 25\%, \frac{3}{8}, 70\%, 4.3\%, 0.81, 0.64\]
You know that 100% means one, or the whole, and 50% means half. You can estimate a lot of percents when using a visual model.

A laptop computer uses an icon of a battery on the toolbar to show how much power is left in the battery. When you glance at the icon, you can get a good estimate of how much battery life remains before you need to recharge the battery.

1. Estimate how much battery power remains by writing the percent under each battery icon.

   a.  
   b.  
   c.  
   d.  
   e.  
   f.  

2. Estimate the shaded part of each circle shown, and write it as a percent.

   a.  
   b.  
   c.  

Are your estimates the same as your partner’s?
3. Estimate the shaded part of each model, and write it as a fraction, a decimal, and a percent. Write the fraction in lowest terms.

4. Describe the strategies that you used to make your estimations.
A benchmark percent is a percent that is commonly used, such as 1%, 5%, 10%, 25%, 50%, and 100%. With fractions and decimals, benchmarks can be used to make estimations. With percents, however, you can use benchmarks to calculate any whole percent of a number.

1. Use the tape diagram to state each relationship.

   a. How is 50% related to 100%?

   b. How is 25% related to 100%? How is 25% related to 50%?

   c. How is 10% related to 100%? How is 10% related to 50%?
2. Continue the pattern from the tape diagram to state each relationship.
   a. How is 5% related to 10%?

   b. How is 1% related to 10%? How is 1% related to 5%?

3. Use the benchmark percents to determine each value if 600 is 100%.
   a. 50%  
   b. 25%
   c. 10%  
   d. 5%  
   e. 1%
4. Use your calculator to determine the percent of each number.
   
a. 1% of 28 =  
b. 10% of 28 =  

c. 1% of 234 =  
d. 10% of 234 =  

e. 1% of 0.85 =  
f. 10% of 0.85 =  

g. 1% of 5.86 =  
h. 10% of 5.86 =  

i. 1% of 98.72 =  
j. 10% of 98.72 =  

k. 1% of 1085.2 =  
l. 10% of 1085.2 =  
5. What patterns do you notice in your answers in Question 4?

6. Write a rule to calculate 1% of any number.

7. Write a rule to calculate 10% of any number.

8. Use the patterns you recognized in Question 4 to calculate each value.
   a. 10% of 45.21
   b. 1% of 45.21
   c. 10% of 0.72
   d. 1% of 0.72
   e. 10% of 2854
   f. 1% of 2854
Deciding how much tip to leave a server at a restaurant is one way that percents are used in the real world.

Akurio eats at the Eat and Talk Restaurant and decides to leave a 15% tip. Akurio says, “I can easily calculate 10% of any number, and then calculate half of that, which is equal to 5%. I can then add those two percent values together to get a sum of 15%.”

1. Is Akurio’s method reasonable?

2. How much should he leave for a tip of 15% on $16.00?

3. What is 15% of each restaurant check total given? Explain how you calculated your answer. Round to the nearest hundredth if necessary.

   a. $24.00   b. $32.56   c. $47.00

You can determine any whole percent of a number by using 10%, 5%, and 1%.

4. How can you use 10%, 5%, and/or 1% to determine each percent given? Explain your reasoning.

   a. 18%   b. 25%   c. 37%
5. Calculate each value using 1%, 5%, and 10%.
   a. 27% of 84        b. 43% of 116
   
c. 98% of 389        d. 77% of 1400
   e. 12% of 1248

6. About 12% of the United States population is left-handed. Use this estimate to determine about how many left-handed students there would be for each class of the given size.
   a. 150 students
   b. 200 students
   c. 375 students

So, if 12 percent of the U.S. population is left-handed, what percent of the population is right-handed or "both"-handed?
TALK the TALK

Brain Weights

A chimpanzee’s brain weight can be compared to the brain weight of other mammals. Assume that the weight of an average chimpanzee’s brain is 400 grams. The table provides the average brain weight of various mammals as a percent of a chimp’s brain weight.

<table>
<thead>
<tr>
<th></th>
<th>Lion</th>
<th>Sheep</th>
<th>Cat</th>
<th>Rabbit</th>
<th>Human</th>
<th>Bear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Brain Weight</td>
<td>60%</td>
<td>35%</td>
<td>7%</td>
<td>2.5%</td>
<td>350%</td>
<td>119%</td>
</tr>
<tr>
<td>as a Percentage of</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a Chimp’s Brain Weight</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average Brain Weight</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(grams)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Order from least to greatest the brain weights of the mammals in the table, along with the chimpanzee, based on percents.

2. Use benchmarks to determine the average brain weights for each animal. Show all of your work.

3. Does the order of the percents match the order of the brain weights? Why or why not?
Write
Explain how to use benchmark percents to order and estimate the value of other percents.

Remember
Benchmarks percents—1%, 5%, 10%, 25%, 50%, and 100%—can be used to perform mental estimation and calculation of percents. Values of benchmark percents can be added and subtracted to calculate the value of other percents.

Practice
The students at Penncrest Middle School sold various products for a fall fundraiser. The table shows the percent of profit the school earned and the total amount sold for each type of product.

<table>
<thead>
<tr>
<th>Product</th>
<th>Percent Profit</th>
<th>Amount Sold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Candy</td>
<td>65%</td>
<td>$6400</td>
</tr>
<tr>
<td>Wrapping paper</td>
<td>40%</td>
<td>$1200</td>
</tr>
<tr>
<td>Stationery</td>
<td>50%</td>
<td>$900</td>
</tr>
<tr>
<td>Calendars</td>
<td>25%</td>
<td>$3120</td>
</tr>
</tbody>
</table>

1. Use benchmark percents to calculate the amount of profit the school earned on the sale of each product.
   a. Candy
   b. Wrapping paper
   c. Stationary
   d. Calendars

2. Suppose that the students also sold $4500 worth of pens and pencils, which earned a 42% profit. Calculate the profit the school earned on pens and pencils.
**Stretch**
Assume the weight of an average chimpanzee’s brain is 400 grams. If the average hedgehog’s brain weight is 0.8% of a chimp’s brain weight, use benchmark percents to determine the average weight of a hedgehog’s brain.

**Review**
1. Complete the table. Write each as a fraction, decimal, and percent.

<table>
<thead>
<tr>
<th>Fraction</th>
<th>Decimal</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3%</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>$\frac{13}{20}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{2}{3}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Miss Jenn is the teacher of a preschool class at Kids Unlimited Daycare. She must split the children’s time between playing and learning. For every 30 minutes, the children will spend 18 minutes playing and 12 minutes learning. Complete the table of equivalent ratios.

<table>
<thead>
<tr>
<th>Total amount of time</th>
<th>30</th>
<th>90</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Playing time</td>
<td>18</td>
<td></td>
<td>144</td>
</tr>
<tr>
<td>Learning time</td>
<td>12</td>
<td></td>
<td>48</td>
</tr>
</tbody>
</table>

3. Use the standard algorithm to determine each quotient.
   a. $8302 \div 28$
   b. $39.13 \div 4.3$
LESSON 3: The Forest for the Trees

The Forest for the Trees
Determining the Part and the Whole in Percent Problems

WARM UP
Complete each equivalent fraction.

1. \( \frac{6}{8} = \frac{\Box}{12} \)
2. \( \frac{\Box}{16} = \frac{15}{40} \)
3. \( \frac{6}{15} = \frac{\Box}{10} \)

LEARNING GOALS
- Use ratio and rate reasoning to solve percent problems involving determining the part, the percent, and the whole.
- Solve percent problems involving determining the part, given the whole and the percent.
- Use ratio reasoning to estimate the value of the whole in percent problems.
- Solve percent problems involving determining the whole.

You have learned how to use benchmarks to determine the percent of a number, but what if you only know the part and the percent? How can you use your knowledge of percents to determine the whole amount?
The Big Picture

When you study problems in terms of ratios, like percents, it is important to think about the whole as well as the parts. The whole is not always 100 or 1. And if the whole changes, this changes the percent!

1. Consider the picture of triangles.
   a. If the picture shown is 100% of the triangles, draw 50% of the triangles.
   b. If the picture shown is 30% of the triangles, draw 100% of the triangles.

2. The given rectangle represents 25% of the whole figure.
   a. Draw a rectangle that represents 50% of the whole figure.
b. Draw a rectangle that represents 75% of the whole figure.

c. Draw a rectangle that represents 100% of the whole figure.

3. The figure shown represents 75% of the whole figure.
   
a. Draw 25% of the figure.

b. Draw 100% of the figure.
Mr. Goodwin, the sixth grade math teacher, asked the class to determine 25% of 44. Five different student responses are shown.

Kendra
Since 25% of 44 means multiplying $\frac{25}{100}$ times the quantity, I used the fraction method.
$\frac{25}{100} = \frac{1}{4}$
Then, I multiply $\frac{1}{4} \cdot 44 = 11$.

Hank
I like decimals much better than fractions.
$\frac{25}{100} = 0.25$
$0.25 \cdot 44 = 11$

Ryan
25% is easy to do in my head. 50% of 44 is 22.
25% is $\frac{1}{2}$ of 50%, so 25% of 44 is $\frac{1}{2}$ of 22, which is equal to 11.

Simon
Since 25% is the same as $\frac{1}{4}$, I just divided by four:
$44 \div 4 = 11$

Pamela
I prefer to use the benchmarks of 10% and 5%.
10% of 44 = 4.4.
20% is 2 • 10% = 2 • 4.4 = 8.8.
5% is half of 10% = 2.2.
Therefore, 20% + 5% = 8.8 + 2.2 = 11.0
1. Discuss each student method used.
   
a. When is Kendra’s method most efficient to use?

b. When is Hank’s method most efficient to use?

c. When is Ryan’s method most efficient to use?

d. When is Simon’s method most efficient to use?

e. When is Pamela’s method most efficient to use?

A more efficient method is one that requires fewer steps or simpler steps to determine an answer.
Ellen said, “All the methods are correct, and everyone got the correct answer, but what if Mr. Goodwin gave us the problem 32% of 732?”

- Kendra said, “My fraction method is not as easy this time.”
  \[
  \frac{32}{100} \times \frac{732}{1} = \frac{5856}{25} = 234.24
  \]

- Hank said, “32% = 0.32
  
  0.32 \times 732 = 234.24
  
  My method is not any more difficult this time.”

- Ryan said, “I can still estimate . . . , but my answer will be close, not exact. 32% is close to \( \frac{1}{3} \) and \( \frac{1}{3} \) of 732 is 244.”

- Simon said, “I don’t have an easy fraction to use for 32%, so my method works only for certain percents.”

- Pamela said, “I can still use my method.”
  
  32% = 10% + 10% + 10% + 1% + 1%
  
  10% of 732 = 73.2
  
  1% of 732 = 7.32
  
  73.2(3) = 219.6
  
  7.32(2) = 14.64
  
  219.6 + 14.64 = 234.24

2. Which method do you prefer with this particular percent of a quantity problem? Explain your thinking.

3. Determine the percent of each quantity.
   a. 7% of 80
   b. 15% of 55
   c. 12% of 320
   d. 8% of 300
Karla is in charge of designing a way to keep a running total of the money raised by her homeroom for the Food Bank project. As of today, her homeroom has raised $240, which is 60% of their goal.

WORKED EXAMPLE

Karla decided to use a double number line to record the money raised and the percent of the goal raised.

The bottom number line represents the percent of the homeroom goal. The top number line represents the amount of money raised.

Karla’s homeroom has raised $240, which is 60% of the goal.
1. How did Karla determine the value that corresponds to 10%?

2. If $240 is 60% of the homeroom goal, what is 100% of the goal? Explain how you determined the answer.

3. Which way of reporting is more informative: the amount of money raised, or the percent of money raised? Explain your thinking.

4. Complete each double number line to represent the goals of the other sixth grade homerooms using the information from the table. Write the equivalent dollar amount for each percent shown.

<table>
<thead>
<tr>
<th>Homeroom</th>
<th>6A</th>
<th>6B</th>
<th>6C</th>
<th>6D</th>
<th>6E</th>
<th>6F</th>
</tr>
</thead>
<tbody>
<tr>
<td>60% of Goal</td>
<td>240</td>
<td>144</td>
<td>288</td>
<td></td>
<td>168</td>
<td></td>
</tr>
<tr>
<td>100% of Goal</td>
<td>400</td>
<td>360</td>
<td>120</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
a. Homeroom 6B

b. Homeroom 6C

c. Homeroom 6D

d. Homeroom 6E

e. Homeroom 6F
You can use proportions to determine the whole in percent problems.

WORKED EXAMPLE

Carlos is told that 65% of the students, or 78 students, prefer pizza for lunch according to a recent survey. He wants to know how many students were surveyed. He drew the model shown to visualize the problem.

He then wrote the proportion and determined that 120 students were surveyed.

\[
\frac{\text{part}}{\text{whole}} = \frac{78}{?} = \frac{65}{100}
\]

\[
\frac{65}{100} = \frac{13}{20} = \frac{78}{?}
\]

\[
\frac{78}{120} = \frac{65}{100}
\]

1. How did Carlos determine the total number? Explain Carlos’ calculations.
2. Use Oscar’s method to determine the unknown value.

\[
\frac{45}{100} = \frac{126}{?}
\]

3. Determine the whole in each situation. Explain your reasoning.

   a. The best player on your school basketball team makes 60% of her free throws. If she scored 90 points in a season on free throws, which are worth one point each, how many free throws did she attempt?

   b. You got a quiz back and your teacher wrote 16, and 80% at the top. How many points was the quiz worth?
c. Sandy made a 30% deposit on the purchase of a computer. She gave the clerk $168. What is the price of the computer?

d. Your friends ate at a restaurant and left a $2.40 tip. They left a 15% tip. What was the cost of their bill before the tip?

---

**Wholes in Problems**

An accountant is reviewing a store’s financial statements. But some of the information is missing. All of the employee names, their bonus percent, and bonus amounts should be listed in the table.

<table>
<thead>
<tr>
<th>Employee Name</th>
<th>Bonus Percent</th>
<th>Bonus Amount</th>
<th>Total Sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kiesha</td>
<td>18%</td>
<td>$540</td>
<td></td>
</tr>
<tr>
<td>Tonya</td>
<td>21%</td>
<td>$3657.14</td>
<td></td>
</tr>
<tr>
<td>Ruth</td>
<td>15%</td>
<td>$650</td>
<td>$3657.14</td>
</tr>
<tr>
<td>Mario</td>
<td>10%</td>
<td>$3250</td>
<td></td>
</tr>
<tr>
<td>Joseph</td>
<td>23%</td>
<td>$678</td>
<td>$3250</td>
</tr>
</tbody>
</table>

1. Help the accountant by determining each employee’s bonus amount or total sales. Complete the table and show your work.
2. Gareth liked Hank’s method for calculating percents of a number and thought it would work for calculating the whole. He said that if you can multiply to determine the part of a whole, maybe you can divide to determine the whole when you know only the part.

He gave this example:

<table>
<thead>
<tr>
<th>Percent</th>
<th>Whole</th>
<th>Part</th>
<th>Unknown Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>20% of</td>
<td>$1500</td>
<td>_____</td>
<td>$0.2 \times 1500</td>
</tr>
<tr>
<td>20% of</td>
<td>_____</td>
<td>$300</td>
<td>$300 \div 0.2</td>
</tr>
</tbody>
</table>

Is Gareth correct? Does this always work? Explain your thinking.

3. Determine each value.

a. 15 is 25% of what number?  

b. 15 is 30% of what number?

c. 45 is 75% of what number?  

d. 16 is 20% of what number?

e. 36 is 40% of what number?  

f. 6 is 15% of what number?

g. 27 is 30% of what number?
4. A department store recently had a big sale where the prices of items were marked 25% off of the regular price. Now that the sale is over, Tremain needs to mark each of the items back up to its original price. The items and their sale prices are listed in the table. Help Tremain complete the table.

<table>
<thead>
<tr>
<th>Item</th>
<th>Sale Price</th>
<th>Original Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>shirt</td>
<td>$24.00</td>
<td></td>
</tr>
<tr>
<td>pants</td>
<td>$36.00</td>
<td></td>
</tr>
<tr>
<td>sweater</td>
<td>$59.95</td>
<td></td>
</tr>
<tr>
<td>suit</td>
<td>$299.00</td>
<td></td>
</tr>
<tr>
<td>sports coat</td>
<td>$159.95</td>
<td></td>
</tr>
</tbody>
</table>

5. The department store realizes it isn’t making enough money. The store manager decides to mark up prices by 20%. Complete the table with the new price for each item.

<table>
<thead>
<tr>
<th>Item</th>
<th>Original Price</th>
<th>New Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>shirts</td>
<td>$22.00</td>
<td></td>
</tr>
<tr>
<td>pants</td>
<td>$29.00</td>
<td></td>
</tr>
<tr>
<td>shoes</td>
<td>$65.00</td>
<td></td>
</tr>
<tr>
<td>jackets</td>
<td>$50.00</td>
<td></td>
</tr>
</tbody>
</table>
6. The department store orders toasters from a company that produces three different models of toasters. The company has found that the percent of each shipment that is defective differs by model. Model A’s defect rate is 2.5%, Model B’s defect rate is 1.75%, and Model C’s defect rate is 3.2%.

On a particular shipment, the company forgets to mark the total number shipped of each model. You only know that you received 5 defective Model A toasters, 7 defective Model B toasters, and 16 defective Model C toasters. How many of each model were shipped?

ACTIVITY 3.5

Wholes in Geometry

You can apply what you have learned about wholes, percents, and ratio reasoning to solve percent problems in geometry too.

1. Corinne’s new dog pen is a rectangular pen that measures 12 yards by 4 yards. She reduced the area of her old rectangular dog pen by 60% after adopting out 6 puppies. List some possible dimensions of Corinne’s old dog pen. Explain your reasoning.
2. The tank shown is 75% full of water.

a. What is the height of the tank? Explain how you solved the problem.

b. Suppose the outside of the tank is covered with paper only up to the water level. What percent of the total surface area of the tank would be covered? Round to the nearest whole percent. Be sure to include the top of the tank in the total.
3. Linda wants to make doggy treats for her dog John Henry. The center 20% of each treat will be peanut butter. The rest will be a biscuit made from a mixture of wheat flour, eggs, and mashed bananas. What is the total volume of the doggy treat, including the peanut butter? What is the volume of just the biscuit?

4. The area of Parallelogram A is 25% of the area of Parallelogram B. What is the height of Parallelogram B? Show your work.
TALK the TALK

Try and Try Again

For each question, demonstrate two different ways to determine the answer.

1. Leah’s goal is to score a 90% on the upcoming science test. If there are 40 questions on the test, how many does Leah need to answer correctly?

Plan a presentation of your 2 solutions. Talk about how they are the same and how they’re different.
## Assignment

### Write

Compare different ways to determine the whole in a percent problem: using double number lines, writing a proportion, and using division.

### Remember

Percent problems often have a part, a percent, and a whole. When you know the part and the percent, you can use a variety of strategies to determine the whole.

## Practice

1. A manager at the department store keeps track of “points” for each employee. Employees earn points by being on time for work and for keeping the department neat. On a particular day, he gives “smile” points for each time an employee smiles at a customer. He recorded the smile points that each employee received, along with the total points for that employee. He had a problem with his computer, though, and some of the entries were deleted. Help the manager complete the table.

<table>
<thead>
<tr>
<th>Employee Name</th>
<th>Smile Points</th>
<th>Percent of Total Points</th>
<th>Total Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Garrett</td>
<td>15</td>
<td>5%</td>
<td></td>
</tr>
<tr>
<td>Ricardo</td>
<td>8%</td>
<td>325</td>
<td></td>
</tr>
<tr>
<td>Brent</td>
<td>6</td>
<td>2%</td>
<td></td>
</tr>
<tr>
<td>Lin</td>
<td>21</td>
<td>6%</td>
<td></td>
</tr>
<tr>
<td>Danielle</td>
<td>45</td>
<td>12%</td>
<td></td>
</tr>
</tbody>
</table>

2. The Music Department of a department store sold 12 jazz CDs last month. Jazz sales during that month made up 2% of the Music Department’s total sales.
   a. Determine the number of CDs that the store sold during that month.
   b. Suppose that the store sells 14 jazz CDs during the next month and the percent of sales from jazz CDs is still 2%. What is the total number of CDs that the store will sell?

3. Calculate each value.
   a. 12 is 20% of what number?  
   b. 28 is 35% of what number?  
   c. 84 is 42% of what number?  
   d. 32 is 80% of what number?  
   e. 35% of 60 is what number?  
   f. 25% of 132 is what number?  
   g. 5% of 40 is what number?  
   h. 15% of 80 is what number?
**Stretch**

Bob ate at a restaurant one night with 2 friends. The cost of his meal was 10% of the cost of the 3 meals before the tip. Four receipts from the restaurant that night are shown. Some of the groups gave an 18% tip and some gave a 15% tip, but you’re not sure which is which.

1. Which guest was Bob, and where was he sitting?

<table>
<thead>
<tr>
<th>TABLE 1</th>
<th>TABLE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guest 1:</td>
<td>Guest 1: $38.45</td>
</tr>
<tr>
<td>Guest 2: $18.00</td>
<td>Guest 2: $34.81</td>
</tr>
<tr>
<td>Guest 3: $12.00</td>
<td>Guest 3:</td>
</tr>
<tr>
<td>Tip: $4.96</td>
<td>Tip: $14.65</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TABLE 3</th>
<th>TABLE 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Guest 1:</td>
<td>Guest 1: $35.11</td>
</tr>
<tr>
<td>Guest 2:</td>
<td>Guest 2: $35.05</td>
</tr>
<tr>
<td>Guest 3: $41.00</td>
<td>Guest 3:</td>
</tr>
<tr>
<td>Tip: $11.74</td>
<td>Tip: $15.90</td>
</tr>
</tbody>
</table>

**Review**

1. Jai has a 28% free throw rate in basketball. That means when he shoots a free throw he makes a basket 28% of the time. Jai shoots 120 free throws in a season. How many baskets is he likely to make? Use benchmark percents of 1% and 10% to help you determine the answer.
   a. What is 1% of 120? 
   b. What is 10% of 120?
   c. What is 20% of 120? 
   d. What is 8% of 120?

2. In Tampa, Florida, the sun shines about 66% of the year. About how many days does the sun shine in Tampa?

3. Bill is painting his room a certain shade of green. The paint is a mixture of 3 parts blue paint to 2 parts yellow paint. To get the correct shade of green, how much yellow paint should he add to 6 quarts of blue paint?

4. LaShaya answered 9 out of 10 questions correctly on her math quiz. Her twin sister LaTeisha answered 22 out of 25 questions correctly on her math test. Did they have the same ratio of correct problems to total problems?

5. Determine each product.
   a. $0.6 \times 95$
   b. $210 \times 0.75$
Percents Summary

KEY TERM
• benchmark percents

We Are Family!

Percent can be used to represent a part-to-whole relationship with a whole of 100. The symbol “%” means “out of 100.” You can think of a percent as a fraction in which the denominator is 100.

Percents, fractions, and decimals can be used interchangeably.

For example, you can write 15 out of 100 as the fraction \( \frac{15}{100} \) or \( \frac{3}{20} \). Written as a decimal, 15 out of 100 is 0.15. Because percent means “out of 100”, 15 out of 100 can also be written as 15%.

When the denominator is a factor of 100, scale up the fraction to write it as a percent.

\[
\begin{align*}
&\frac{4}{5} = \frac{80}{100} \\
&\times 20 \\
&80 \div 100 = 80\
\end{align*}
\]

When the denominator is not a factor of 100, you can divide the numerator by the denominator to write the fraction as a decimal, which you can then write as a percent.

\[
\begin{align*}
0.625 &= 8 \div 12.500 \\
&= 8 \div 12.500 \\
&= 0.5 \\
&= 80 \\
&= 0.625 \\
&= 62.5% \\
\end{align*}
\]
When ordering numbers expressed as fractions, decimals, and percents, you can first write the numbers in the same form before comparing.

For example, to order the numbers 0.88, 90%, and \( \frac{17}{20} \) from least to greatest, you can write each number as a percent.

\[
0.88 = \frac{88}{100} = 88\%
\]
\[
\frac{17}{20} = \frac{85}{100} = 85\%
\]

The numbers in order from least to greatest are \( \frac{17}{20} \), 0.88, and 90%.

You can estimate percents when using a visual model.

For example, the shaded part appears to be about \( \frac{1}{3} \) of the whole circle, and
\[
\frac{1}{3} \approx 33\%.
\]

A benchmark percent is a percent that is commonly used, such as 1%, 5%, 10%, 25%, 50%, and 100%. With fractions and decimals, benchmarks can be used to make estimations. With percents, however, you can use benchmarks to calculate any whole percent of a number.
For example, determine each value if 400 is 100%.
There is more than one way to use benchmark percents to determine the values.

You can determine any whole percent of a number by using 10%, 5%, and 1%.

For example, what is 28% of 500?

\[
28\% = 10\% + 10\% + 5\% + 1\% + 1\% + 1\% + 1\% + 1\% + 1\% + 1\%
\]

10% of 500 is \(500 \times \frac{1}{10}\), or 50.

5% of 500 is \(50 \times \frac{1}{20}\), or 25.

1% of 500 is \(25 \times \frac{1}{50}\), or 5.

\[50 + 50 + 25 + 5 + 5 + 5 + 5 + 5 + 5 + 5 = 140\]

28% of 500 is 140.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 50%</td>
<td>50% is half of 100%.</td>
</tr>
<tr>
<td>500 \times \frac{1}{2}</td>
<td>200</td>
</tr>
<tr>
<td>b. 25%</td>
<td>25% is half of 50%.</td>
</tr>
<tr>
<td>200 \times \frac{1}{2}</td>
<td>100</td>
</tr>
<tr>
<td>c. 10%</td>
<td>10% is one-fifth of 50%.</td>
</tr>
<tr>
<td>200 \times \frac{1}{5}</td>
<td>40</td>
</tr>
<tr>
<td>d. 5%</td>
<td>5% is half of 10%.</td>
</tr>
<tr>
<td>40 \times \frac{1}{2}</td>
<td>20</td>
</tr>
<tr>
<td>e. 1%</td>
<td>1% is one-fifth of 5%.</td>
</tr>
<tr>
<td>20 \times \frac{1}{5}</td>
<td>4</td>
</tr>
</tbody>
</table>

Percent problems often have a part, a percent, and a whole. When you know the part and the percent, you can use a variety of strategies to determine the whole.

One strategy is a double number line.

For example, Karla’s homeroom raised $240 for charity, which is 60% of their goal. Karla uses a double number line to record the amount of money raised and the percent of the goal raised.

![Double Number Line Diagram]

**TOPIC 1: SUMMARY** • M2-159
Karla’s homeroom has raised $240, which is 60% of the goal.

To determine the value that corresponds to 10%, Karla divided the amount raised so far by 6: 
$240 \div 6 = $40.

Since 10% × 10 = 100%, she can multiply $40 by 10 to determine the homeroom’s goal:  
$40 × 10 = $400.

You can also use proportions to determine the whole in percent problems.

For example, Carlos is told that 65% of the students, or 78 students, prefer pizza for lunch according to a recent survey. He wants to know how many students were surveyed.

He wrote a proportion and determined that 120 students were surveyed.

These strategies can be used to solve geometry problems as well.

For example, the tank shown is 75% full of water. What is the height of the tank?

The volume of the water can be calculated using the formula \(V = Bh\) where the \(B\) is equal to the area of the base, and \(h\) is equal to the height of the water in the tank.

Volume of water = 5.75 × 7.5 × 8 = 345 cubic inches

The volume of 345 cubic inches is 75% the volume of the whole tank. Set up a proportion and scale up to determine the volume of the tank.

\[
\frac{75}{100} = \frac{345}{?} \times 4.6
\]

The volume of the tank is 460 cubic inches.

Divide the volume of the tank by the area of its base to determine the tank’s height.

\[
460 \div 43.125 = 10.67 \text{ inches}
\]
Most car models are sold all over the world, not just in the United States, so their speedometers show both miles per hour (mph) and kilometers per hour (km/h).

**Lesson 1**
Many Ways to Measure
Using Ratio Reasoning to Convert Units ........................................... M2-165

**Lesson 2**
What Is the Best Buy?
Introduction to Unit Rates ................................................................. M2-185

**Lesson 3**
Seeing Things Differently
Multiple Representations of Unit Rates .............................................. M2-199
TOPIC 3: UNIT RATES AND CONVERSIONS

Students learn that converting within and between systems of measurement involves the use of conversion rates, another special type of ratio. To convert units of measurement, they use double number lines, ratio tables, scaling up or down, and unit analysis. Students use models to illustrate the meaning of a unit rate, with each quantity as the denominator. They solve a variety of unit rate problems, determining which unit rates make sense in the context of a problem. Students evaluate prices to determine the better buy and solve problems involving constant speed. Finally, they analyze scenarios and clearly identify the unit rates from tables and graphs.

Where have we been?
Students enter grade 6 with experience converting among different-sized standard units within a given system of measurement. Students use strategies from the previous topics – tables and double number lines – to complete conversions, moving from multiplicative reasoning to ratio strategies to unit analysis. As students continue in the topic, they use all of the strategies developed in the previous topic to solve unit rate problems.

Where are we going?
This topic provides the foundation for important ideas in algebra and science: slope and dimensional analysis. In grade 7, students will use their understanding of unit rate to represent proportional relationships between quantities. They will use unit rate to write equations and graph proportional relationships, developing an informal understanding of slope.

Using a Coordinate Plane to Visualize Unit Rates

Unit rates can be graphed on a coordinate plane. For example, this graph shows the unit rate $7.50 per item, which is approximately 0.133 item per dollar. Moving up and down the line and reading the coordinates will give you equivalent rates.
Myth: Just watch a video, and you will understand it.
Has this ever happened to you? Someone explains something, and it all makes sense at the time. You feel like you get it. But then, a day later when you try to do it on your own, you suddenly feel like something’s missing? If that feeling is familiar, don’t worry. It happens to us all. It’s called the illusion of explanatory depth, and it frequently happens after watching a video.

How do you break this illusion? The first step is to try to make the video interactive. Don’t treat it like a TV show. Instead, pause the video and try to explain it to yourself or to a friend. Alternatively, attempt the steps in the video on your own and rewatch it if you hit a wall. Remember, it’s easy to confuse familiarity with understanding.

#mathmythbusted

Talking Points
You can further support your student’s learning by asking questions about the work they do in class or at home. Your student is learning to work with unit rates and measurement conversions.

Some Things to Look For
Look for real-life examples of conversions, like inches to feet, days to years. Any time you ask about a measurement in different units, you’ll need to do a conversion with ratios and rates. When your student converts a measurement to different units, ask them to explain whether their answer makes sense.

Key Term
**unit rate**
A unit rate is a comparison of two measurements in which the denominator has a value of one unit.
LESSON 1: Many Ways to Measure

Using Ratio Reasoning to Convert Units

WARM UP
Answer each question about a common measurement conversion.
1. How many inches are in 1 foot?
2. How many feet are in 1 yard?
3. How many grams are in 1 kilogram?
4. How many milliliters are in 1 liter?
5. How many centimeters are in 1 meter?
6. How many fluid ounces are in 1 cup?
7. How many quarts are in 1 gallon?
8. Which units in Questions 1–7 are part of the U.S. customary system of measurement and which are part of the metric system?

LEARNING GOALS
• Use ratio reasoning with double number lines to convert measurement units.
• Use ratio reasoning with ratio tables to convert measurement units.
• Use scaling up or scaling down to convert and transform measurement units appropriately.
• Use unit analysis to convert and transform measurement units appropriately.

KEY TERM
• convert

In previous grades, you have worked with the U.S. customary system and the metric system of measurement. This year, you have also learned about ratios. How can you use ratio reasoning to convert from one measurement unit to another in order to solve problems?
Customary to Whom?

In the U.S., customary units are primarily used for business, personal, and social purposes. Sciences, including the medical field, use the metric system.

You’ve learned about the relationships between inches and feet, feet and yards, quarts and gallons, meters and millimeters—to name a few.

1. Name a U.S. customary system unit and a metric system unit that would be an appropriate size to measure each object or quantity.

<table>
<thead>
<tr>
<th>Object/Quantity</th>
<th>U.S. Customary System</th>
<th>Metric System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Your height</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Length of your pencil</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Distance from your school to the beach</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Weight of your math book</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Amount of water in a bottle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Amount of water in a swimming pool</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Circle the most appropriate measurement for each item.

a. The weight of a dog
   - 15 pounds
   - 18 ounces
   - 1 ton
   - 25 fluid ounces

b. The amount of gas in a car’s tank
   - 50 milliliters
   - 2 kiloliters
   - 55 liters
   - 12 kiloliters

c. The height of your classroom
   - 90 inches
   - 1 mile
   - 2 yards
   - 12 feet

d. The height of a basketball hoop
   - 3 meters
   - 70 centimeters
   - 500 millimeters
   - 1 kilometer
You can use more than one measurement to describe the same length, weight, or capacity. For example, you may say that a football field is 100 yards long or 300 feet long. You could also say that the football field is about 90 meters long. In each case, the lengths are the same—you just say them in different ways.

There are many situations in which you need to convert measurements to different units. To convert a measurement means to change it to an equivalent measurement in different units.

1. Name a situation in which converting one measurement to another would be necessary or useful.

Before you start converting units, it is useful to estimate the number of units to expect in a conversion. A few estimates comparing common metric and U.S. customary measures are given.

- One meter is about the same length as one yard.
- One inch is about 2.5 centimeters.
- One kilometer is a little more than half of a mile.
- One foot is about 30 centimeters.
- One liter is about the same as one quart.
- One kilogram is a little more than 2 pounds.
Use the estimates given and your knowledge of metric and U.S. customary measures to answer each question.

2. The numeric value of which measurement will be greater?
   a. The length of a table in inches or in feet
   b. The length of a table in meters or in centimeters
   c. The length of a table in meters or in yards
   d. The distance from school to your house in miles or in kilometers
   e. The weight of your math book in kilograms or in pounds

3. How did you decide which value would be greater in Question 2?

4. Estimate each measurement conversion.
   a. The distance to Toronto is 548 km. About how many miles is that?
   b. You order 5 kilograms of food pellets for your guinea pig. About how many pounds are you ordering?

5. Describe the strategies you used to estimate each measurement conversion in Question 4.
Because most conversions compare two quantities using multiplicative strategies, the conversion estimates provided and the conversions within systems that you already know can be written using ratio language. They can also be written symbolically in terms of equality.

<table>
<thead>
<tr>
<th>Ratio Language</th>
<th>Symbolically</th>
</tr>
</thead>
<tbody>
<tr>
<td>For every inch, there are approximately 2.5 centimeters.</td>
<td>1 in. (\approx) 2.5 cm</td>
</tr>
<tr>
<td>For every meter, there is approximately 1 yard.</td>
<td>1 m (\approx) 1 yd</td>
</tr>
<tr>
<td>For every foot, there are approximately 30 centimeters.</td>
<td>1 ft (\approx) 30 cm</td>
</tr>
<tr>
<td>For every 12 inches, there is exactly 1 foot.</td>
<td>12 in. = 1 ft</td>
</tr>
<tr>
<td>For every 1 kilometer, there are exactly 1000 meters.</td>
<td>1 km = 1000 m</td>
</tr>
</tbody>
</table>

When a conversion ratio is presented for use in converting between units of measure, it is often written as an equation: 12 in. \(=\) 1 ft. However, it can also be written as a ratio in fractional form: \(\frac{12 \text{ in.}}{1 \text{ ft}}\).

6. Rewrite each common conversion using ratio language and as a ratio in fractional form.

   a. 3 ft \(=\) 1 yd   b. 5280 ft \(=\) 1 mi

   c. 1 lb \(\approx\) 0.45 kg   d. 4 qt \(=\) 1 gal

   e. 1 m = 100 cm   f. \(\frac{1}{1000}\) m = 1 mm

Because these measurement conversion are ratios, you can use ratio reasoning to convert between units. For example, you can determine the number of miles in a 10-kilometer race or the number of fluid ounces in 500 milliliters of a solution.
When you learned about ratios, you learned how to use double number lines to determine equivalent ratios. You can also use double number lines to convert from one unit to another.

Although you may not have realized it before, many rulers are set up as double number lines and can be used to convert between inches and centimeters.

1. Determine which scale represents inches and which represents centimeters. How did you decide? Label the scales on the ruler.

2. Use the ruler as a double number line to determine each approximate conversion.
   a. $1 \text{ cm} \approx \underline{\quad} \text{ in.}$
   b. $1 \text{ in.} \approx \underline{\quad} \text{ cm}$
   c. $5 \text{ cm} \approx \underline{\quad} \text{ in.}$
   d. $3 \text{ in.} \approx \underline{\quad} \text{ cm}$
You are baking cookies at your friend’s house. After searching the cupboards and drawers, you cannot find the measuring cups, but you can find the tablespoon.

3. Use the double number line to determine how many tablespoons you need of each ingredient in the recipe.

```
Tablespoons
0   16
Cups
0   1
```

a. 2 cups of sugar

b. $1 \frac{3}{4}$ cups of flour

c. $\frac{1}{2}$ cup of raisins

4. Suppose you had found the cup but not the tablespoon. Use the double number line to determine how many cups you need if the recipe calls for 2 tablespoons of vanilla extract.
You want to redecorate your bedroom and need to measure the room for new carpeting, paint, and a border on the walls. You realize that you have only a meter stick. You measure the room, but you need to know the dimensions in feet to purchase the materials. You record these measurements:

- The length of the room is 5 meters.
- The width of the room is 4 meters.
- The height of the room is 2.5 meters.

5. Use a double number line to determine the measurement of each dimension in feet.
   a. length  
   b. width  
   c. height

You can use ratio tables, as you did when determining equivalent ratios, as another strategy for converting units.

1. Complete the ratio table by converting between pounds and ounces.

<table>
<thead>
<tr>
<th>Pounds</th>
<th>1</th>
<th>2</th>
<th>1 1/4</th>
<th>1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ounces</td>
<td>16</td>
<td>4</td>
<td>6</td>
<td>40</td>
</tr>
</tbody>
</table>

2. What strategies did you use to determine the missing values?
3. Complete the ratio table by converting between milliliters and liters.

<table>
<thead>
<tr>
<th>Milliliters</th>
<th>1000</th>
<th>100</th>
<th>50</th>
<th>1</th>
<th>575</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liters</td>
<td>1</td>
<td>0.5</td>
<td></td>
<td></td>
<td>0.01</td>
</tr>
</tbody>
</table>

4. What strategies did you use to determine the unknown values?

Ratio tables are nice tools for converting within a given system of measurement. Scaling up or down is a similar strategy for determining equivalent ratios that can be more easily used to convert from one unit of measurement to another.

You will use the common conversions shown in the table to convert between customary and metric systems.

<table>
<thead>
<tr>
<th>Length</th>
<th>Mass</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 in. = 2.54 cm</td>
<td>1 oz = 28.35 g</td>
<td>1 pt = 0.47 L</td>
</tr>
<tr>
<td>1 cm = 0.39 in.</td>
<td>1 g = 0.035 oz</td>
<td>1 L = 2.11 pt</td>
</tr>
<tr>
<td>1 ft = 30.48 cm</td>
<td>1 lb = 0.45 kg</td>
<td>1 qt = 0.95 L</td>
</tr>
<tr>
<td>1 m = 3.28 ft</td>
<td>1 kg = 2.2 lb</td>
<td>1 L = 1.06 qt</td>
</tr>
<tr>
<td>1 mi = 1.61 km</td>
<td></td>
<td>1 gal = 3.79 L</td>
</tr>
<tr>
<td>1 km = 0.62 mi</td>
<td></td>
<td>1 L = 0.26 gal</td>
</tr>
<tr>
<td>1 m = 39.37 in.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 in. = 0.0254 m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 m = 1.09 yd</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Most conversions that require moving between the U.S. customary and metric systems are approximations, so, in general, you will use conversion rates rounded to the nearest hundredth in your calculations.
Scaling up or down is another strategy that you already know that can be used to convert between units.

**WORKED EXAMPLE**

You can use scaling up to determine how many kilograms are in 2.5 pounds. Because you want to determine the number of kilograms for a specific number of pounds, use the conversion rate $1 \text{ lb} = 0.45 \text{ kg}$ or $\frac{1 \text{ lb}}{0.45 \text{ kg}}$.

\[
\frac{1 \text{ lb}}{0.45 \text{ kg}} = \frac{2.5 \text{ lb}}{? \text{ kg}} \quad \rightarrow \quad \frac{1 \text{ lb}}{0.45 \text{ kg}} \times 2.5 = \frac{2.5 \text{ lb}}{1.125 \text{ kg}}
\]

5. Why was the conversion rate $\frac{1 \text{ lb}}{0.45 \text{ kg}}$ used rather than the rate $\frac{2.2 \text{ lb}}{1 \text{ kg}}$?

Use scaling up or down to answer each question.

6. The school cafeteria has eight very large cans of tomato sauce for making pizza. Each can contains 2 gallons of sauce. Is there more or less than 50 L of sauce in these 8 cans?
7. Tyrone, the quarterback for the Tigers Football team, can throw a football 40 meters. Jason, the quarterback for the Spartans, can throw a football 45 yards. Who can throw farther? How do you know?

8. Molly says that she is 1.5 meters tall. Shawna is 5 feet tall. Molly says that she is taller, but Shawna disagrees. Who is correct? Explain your reasoning.

9. Larry weighs 110 pounds, Casey weighs 98 pounds, Shaun weighs 42 kg, and Jamal weighs 52 kg. Place the boys in order from the least weight to the greatest weight using pounds and kilograms.

10. Karen has a gold bracelet that weighs 24 grams. She wants to sell the bracelet, but she needs a minimum of one ounce of gold to sell it. Can Karen sell her bracelet? Why or why not?
To use scaling up or down to convert one unit to another, you set up a proportion and use the conversion rate based on the given measurement that you are converting. In another strategy, unit analysis, you are multiplying by a form of 1 to rewrite the given measurement in a different unit.

**WORKED EXAMPLE**

Determine the quantity in pounds that is equivalent to 4.5 kilograms.

**Scaling Up**

\[
\begin{align*}
1 \text{ kg} & \quad \times 4.5 \\
2.2 \text{ lb} & = 4.5 \text{ kg} \\
\times 4.5 & = ? \text{ lb}
\end{align*}
\]

**Unit Analysis**

\[
\begin{align*}
4.5 \text{ kg} \left( \frac{2.2 \text{ lb}}{1 \text{ kg}} \right) & = 9.9 \text{ lb} \\
1 \text{ kg} & = \frac{4.5 \text{ kg}}{2.2 \text{ lb}} \\
\frac{4.5 \text{ kg}}{2.2 \text{ lb}} & = 9.9 \text{ lb}
\end{align*}
\]

1. Analyze the worked examples.

   a. Both strategies used a form of 1 to determine the equivalent number of pounds in 4.5 kilograms. How is the form of 1 used in scaling up different from the form of 1 used in unit analysis?

   b. Why are the labels for kilograms crossed out in the unit analysis strategy?
Christopher and Max want to determine the number of miles in 31,680 feet using unit analysis.

Christopher

\[
\frac{31,680 \text{ ft}}{} \times \frac{5,280 \text{ ft}}{1 \text{ mi}} = 167,270,400 \text{ mi}
\]

Max

\[
\frac{31,680 \text{ ft}}{} \times \frac{1 \text{ mi}}{5,280 \text{ ft}} = 6 \text{ mi}
\]

2. Explain why Christopher’s answer is not reasonable.

3. Explain what is different in how Christopher and Max set up their multiplication problem. What is important about how the units are arranged in the conversion rates?

Use unit analysis to convert each unit of measurement. Check to make sure your answer is reasonable.

4. A giraffe is 18 feet tall. How tall is the giraffe in inches?
5. A giraffe is 174 inches tall. How tall is the giraffe in feet?

6. The length of the school playground is 32 yards. How many feet long is the playground?

A marathon is a long-distance foot race with an official distance of 42.195 kilometers (26 miles and 385 yards) that is usually run as a road race. Larger marathons can have tens of thousands of runners. Most of these marathon runners are not professional marathoners but run to raise funds for various charities.

7. Although a marathon is a popular distance for a race, there are many other distances in which runners can train to race. Complete the table shown by writing the unknown measurements.

<table>
<thead>
<tr>
<th>Race</th>
<th>Kilometers</th>
<th>Miles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short Distance</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Medium Distance</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Medium Distance</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Half Marathon</td>
<td></td>
<td>13.1</td>
</tr>
<tr>
<td>Ultramarathon</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>Ironman Triathlon Swim</td>
<td></td>
<td>2.4</td>
</tr>
<tr>
<td>Ironman Triathlon Bike</td>
<td></td>
<td>112</td>
</tr>
</tbody>
</table>
Conversion rates are also common in other contexts, like currency. During the 2016 Summer Olympics, the currency exchange rate between the U.S. dollar and the Brazilian real (pronounced “ray-all”) was $1 US for every 3.17 BRL.

8. Alejandra’s family went to the Rio de Janeiro Olympics and she budgeted $500 to spend while she was gone.
   
a. Write the conversion rate: \[ \frac{\text{US}}{} = \frac{\text{BRL}}{} \].
   
b. Did Alejandra budget more or less than 500 BRL? Explain.
   
c. How many BRL could she spend in Rio de Janeiro?
   
d. After Rio de Janeiro, Alejandra’s family traveled to Mexico, where 1 BRL was equal to 5.92 pesos. If Alejandra had 295 BRL remaining, how many pesos did she have?
9. Emma is preparing to re-carpet her room. She measured the room to be 6 yards long and 8 yards wide. When she got to the carpet store, all of the measurements were in square feet.

a. Determine how many square yards of carpet Emma needs to buy to re-carpet her room.

b. Determine how many square feet of carpet Emma needs to buy to re-carpet her room. How can you check your answer?
TALK the TALK

Larger or Smaller?

1. Compare the two conversions. How are they similar? How are they different?

2. When you convert a measurement with smaller units to a measurement with larger units, does the number of units increase or decrease?

3. When you convert a measurement with larger units to a measurement with smaller units, does the number of units increase or decrease?
4. What information is always needed to convert between measurement units?

For each conversion, explain which strategy you prefer to use and then convert the units.

5. 12 gal = _________ L

6. 240 oz = _________ lb

7. 0.380 km = _________ m

8. 324 in = _________ yd
Assignment

Write
Explain how to convert from one unit to another using ratio reasoning.

Remember
More than one unit can be used to describe the same length, weight, or capacity. To convert units means to change a measurement to an equivalent measurement in different units. You can use models, ratio reasoning, and unit analysis to convert units using conversion rates.

Practice
Use any strategy to convert between the specified units.
1. Janine will be traveling to Botswana, where the unit of currency is called the pula, which means “rain” in the local language. Suppose, $1 is equivalent to 7 pula.
   a. If Janine has $500 to spend in Botswana, how many pula will she have to spend?
   b. The safari lodge where she is staying in Chobe National Park costs 434 pula each night. What is the cost per night in dollars?
   c. When she goes to dinner at the safari lodge, the bill comes to 91 pula. How many dollars did Janine spend on dinner?
2. Jonah is going to the hardware store for his Uncle Frederick. He needs to buy 4 yards of electrical wire and 14 quarts of liquid nails.
   a. The store only sells wire by the foot. How many feet does Jonah need?
   b. The store only sells liquid nails by the gallon. How many gallons does Jonah need?
3. Jin Lee is volunteering at a zoo and is helping weigh a penguin’s egg. The egg weighs 0.15 kilogram.
   a. Is this more or less than the average weight of 145 grams? Explain.
   b. If Jin Lee expands the penguin area to be about 500 meters wider than it is now, how many more kilometers wide is the area?
4. Harold is buying a new car. Some of the cars he has researched provide measurements in the U.S. customary system and some provide measurements in the metric system.
   a. One car manufacturer reports the mass of the car to be 3307 lb. How many kilograms is this?
   b. Another manufacturer recommends that the owner change the oil every 12,075 kilometers. After how many miles should the owner change the oil?
   c. Harold is a tall man and prefers cars with high ceilings. One car lists 43.3 inches of headroom and another car lists 99.3 centimeters of headroom. Which car has more headroom?
   d. He is concerned about the fuel tank capacity of the new car he wants to buy. He commutes a long distance to work every day and does not want to constantly be filling the tank. He finds 3 cars that he likes online. The Skye has a fuel capacity of 19 gallons. The Madrid has a fuel capacity of 64.4 liters, and the Cougar has a fuel capacity of 63.6 quarts. Compare the fuel tank capacities of the cars using both gallons and liters. Order the cars from least to greatest fuel tank capacity.
5. A group of 4 campers must navigate through the forest using compasses, topographic maps, and other devices. They scatter and each of them travels to a different location. Using the clues below, determine how far it is from the start to each point on the map.
   - The distance to point A is 1.5 kilometers.
   - It is 0.5 more miles to get to point B from the start than to point A.
   - The total distance to points A and D from the start is 3.1 miles.
   - The distance from the start to point C is twice the distance from the start to point B.
   a. How many kilometers is it from the start to each location?
   b. How many miles is it from the start to each location?

6. A zip line activity is part of an obstacle course that a group of students must get through together. There are several zip lines on the course, the longest of which is about 72 meters long. How can this be stated using the most appropriate unit in the customary system? Show your work.

Stretch

Anthony measured the dimensions of a rectangular box to be 45 cm by 35 cm by 2 m.
1. Determine the volume of the box in cubic meters.
2. Convert the volume of the box to cubic centimeters.

Review

1. At Union Middle School, 99 girls, or 33% of the girls, play basketball. How many girls attend Union Middle School?
2. Kasey gets a 35% employee discount on anything she buys at The Foot Parade. If Kasey got a $5.25 discount on her new flip-flops, how much did they cost originally?
3. Mr. Hawkins manages a small store called Action Sporting Goods. He wants to make sure that his store is stocked with enough equipment for all of the community sports. He surveys 240 of his customers and asks them to choose the one sport that they’re most likely to buy sports equipment for this season.

<table>
<thead>
<tr>
<th>Sport</th>
<th>Percent of Responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basketball</td>
<td>30%</td>
</tr>
<tr>
<td>Baseball</td>
<td>20%</td>
</tr>
<tr>
<td>Football</td>
<td>35%</td>
</tr>
<tr>
<td>Wrestling</td>
<td>15%</td>
</tr>
</tbody>
</table>

   a. How many of the surveyed customers will need baseball equipment?
   b. How many of the surveyed customers will need wrestling equipment?

4. Estimate each quotient to the nearest whole number. Then calculate the quotient.
   a. $0.796 \div 9.95$
   b. $23.84 \div 6.4$
What Is the Best Buy?

Introduction to Unit Rates

WARM UP
Determine each unknown quantity.

1. \[ \frac{18 \text{ arrows}}{3 \text{ bows}} = \frac{162 \text{ arrows}}{? \text{ bows}} \]
2. \[ \frac{18 \text{ arrows}}{3 \text{ bows}} = \frac{? \text{ arrows}}{1 \text{ bow}} \]
3. \[ \frac{8 \text{ shoes}}{80 \text{ socks}} = \frac{? \text{ shoes}}{1600 \text{ socks}} \]
4. \[ \frac{8 \text{ shoes}}{80 \text{ socks}} = \frac{1 \text{ shoe}}{? \text{ socks}} \]

LEARNING GOALS
- Write unit rates.
- Use unit rates to solve problems involving unit pricing and better buys.
- Use unit rates and unit rate language to make comparisons.
- Use unit rates to solve problems involving constant speeds.
- Calculate unit rates.

KEY TERM
- unit rate

Ratios and rates are useful in a variety of real-world situations. Most of your previous work with ratios involved writing equivalent ratios, but ratios, specifically unit rates, can be used to answer many different types of questions. How can unit rates be used in comparisons and to determine which deal is a better buy?
Getting Started

Which One Would You Buy?

Marta and Brad go to the store to buy some laundry detergent for a neighbor. They see that the brand she wants comes in two different sizes:

- 26 fluid ounces for $9.75
- 20.5 fluid ounces for $7.25

1. Which size should Marta and Brad buy? Explain the reason for your decision.
As you learned previously, a rate is a ratio in which the two quantities being compared are measured in different units. A **unit rate** is a comparison of two measurements in which the numerator or denominator has a value of one unit.

One way to compare the values of items is to calculate the unit rate for each item.

Marta estimated unit rates for two detergents this way:

**Marta**

The larger bottle of detergent is about 25 fluid ounces for about $10.

\[
\frac{1 \text{ fl oz}}{25 \text{ fl oz}} = \frac{\$10}{\text{25 fl oz}}
\]

So, each fluid ounce costs about \( \frac{\$10}{25 \text{ fl oz}} \), which is \( \frac{\$2}{5 \text{ fl oz}} \), or \( \frac{\$0.40}{1 \text{ fl oz}} \).

The smaller bottle of detergent is about 21 fluid ounces for about $7.

\[
\frac{1 \text{ fl oz}}{21 \text{ fl oz}} = \frac{\$7}{21 \text{ fl oz}}
\]

So, each fluid ounce costs about \( \frac{\$7}{21 \text{ fl oz}} \), which is \( \frac{\$1}{3 \text{ fl oz}} \), or about \( \frac{\$0.33}{1 \text{ fl oz}} \).

That means that you pay less for each fluid ounce of the smaller bottle of detergent, so it is the better buy.
Brad estimated the unit rates this way:

**Brad**

For the larger bottle of detergent, you spend about $10 for about 25 fluid ounces.

So, for each dollar you spend on the larger bottle of detergent, you get about \( \frac{25 \text{ fl oz}}{\$10} \), or \( \frac{2.5 \text{ fl oz}}{\$1} \).

For the smaller bottle of detergent, you spend about $7 for about 21 fluid ounces.

So, for each dollar you spend on the smaller one, you get about \( \frac{21 \text{ fl oz}}{\$7} \), or \( \frac{3 \text{ fl oz}}{\$1} \).

Because you get more detergent in the smaller bottle for each dollar you spend, the smaller bottle is the better buy.

1. Marta and Brad both chose the smaller bottle of detergent as the better buy, but for different reasons. Explain the differences in their reasoning.

2. Calculate the actual unit rate for each of the two sizes of detergent in two different ways.
Unit rates can be written with either quantity as the unit.

1. Each situation relates a quantity and a price. Calculate the two different unit rates associated with each situation: price per item and number of items per dollar.

   a. A bottle of 250 vitamins costs $12.50.

   b. A pack of 40 AAA batteries costs $25.95.


   d. A box of 500 business cards costs $19.95.

2. Not all unit rates involve money. Write two different unit rates associated with each situation.

   a. The 5 goats eat 12 tomatillos.

   b. The exchange rate is 10 U.S. dollars for every 9 euros.

   c. The average stalactite grows 30 mm every 10 years.

   d. Sandy buys 500 coffee pods every year.

3. For each part of Question 2, identify which unit rates are useful in discussing the situation.
The unit rate needed to solve a problem is often asked for in the question.

4. For each situation, identify the unit rate that would answer the question. Explain how you decided which unit rate to write.

a. How many tomatillos did each goat eat?

b. About how many euros is each U.S. dollar worth?

c. How much does each stalactite grow in a month?

d. How many coffee pods can Sandy use each week?

**ACTIVITY 2.3 Using Unit Rates to Determine the Best Buy**

Movie theater popcorn is sold in notoriously large quantities. The smallest size popcorn usually contains at least 2 servings of popcorn. And, when you’re eating all of that popcorn, you have to get a drink!

1. Compare the prices for various sizes of popcorn sold at the local movie theater.

<table>
<thead>
<tr>
<th>Bag Size</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mega Bag (32 oz)</td>
<td>$10.24</td>
</tr>
<tr>
<td>Giant Bag (24 oz)</td>
<td>$6.00</td>
</tr>
<tr>
<td>Medium Bag (16 oz)</td>
<td>$4.48</td>
</tr>
<tr>
<td>Kid’s Bag (8 oz)</td>
<td>$2.40</td>
</tr>
</tbody>
</table>

a. What is the unit rate price per ounce for each bag of popcorn?
b. What size popcorn is the best buy? Explain your reasoning.

2. Bottles of water are sold at various prices and in various sizes. Write the price of each bottle as a ratio, and then as a unit rate. Which bottle is the best buy? Explain how you know.

3. Use unit rates to determine which is the better buy. Explain your reasoning.
   a. 22 vitamins for $1.97 or 40 vitamins for $3.25
   b. 24.3 ounces for $8.76 or 32.6 ounces for $16.95

4. On a recent trip to the state fair, you saw a sign for the price of the ring toss.
   Which “deal” should you take? Explain your reasoning.
Using Unit Rates to Make Comparisons

The local paper published these rates on gas mileage for a few new cars.

Avalar can travel 480 miles on 10 gallons of gas.
Sentar can travel 400 miles on 8 gallons of gas.
Comstar can travel 360 miles on 9 gallons of gas.

1. Change each rate to a unit rate so that it reports miles per one gallon of gas.
   a. Avalar
   b. Sentar
   c. Comstar

2. How did you calculate each unit rate?

3. How can unit rates help you to compare these cars?

4. Guests at a dinner play are seated at three tables. Each table is served large, round loaves of bread instead of individual rolls. Each person at the table shares the loaves equally.
   Table 1 has six guests and is served two loaves of bread.
   Table 2 has eight guests and is served three loaves of bread.
   Table 3 has 10 guests and is served four loaves of bread.
a. Predict at which table the guests will get the largest serving of bread.

b. Determine how much bread each guest at each table will receive. Was your prediction accurate?

5. Kalida can run 3 laps in 9 minutes. Sonya can run 2 laps in 7 minutes. Who is the faster runner?

6. Peter and Kyu are making mini-cakes for the school bake sale. Peter makes 5 mini-cakes every 25 minutes. Kyu makes 3 mini-cakes every 10 minutes. If they both continue to make mini-cakes at the same rate for the same amount of time, which boy will make more cakes?

7. On Monday, the school cafeteria sold 4 chocolate milks for every 10 white milks. On Tuesday, the cafeteria sold 1 chocolate milk for every 3 white milks. On which day did the cafeteria sell more chocolate milks per number of white milks sold?

8. A tour bus drove 120 miles in 2 hours, and a school bus drove 180 miles in 3 hours. Which bus drove faster?
Unit rates are helpful when solving problems about constant speeds.

1. In the spring, the gym teachers at Stewart Middle School sponsor a bike-a-thon to raise money for new sporting equipment. Students seek sponsors to pledge a dollar amount for each mile they ride.

   a. Nico can ride 12.5 miles per hour. At this rate, how far will he ride in 5 hours?

   b. Grace can ride 14.75 miles per hour. At this rate, how far will she ride in 6 hours?

   c. If Leticia rides 56.25 miles in 5 hours, how far will she ride in 7 hours?

   d. Emil got a cramp in his leg after riding 27.5 miles in 2 hours and had to stop. If he hadn’t gotten the cramp and had continued to ride at the same rate, how far would he have ridden in 3 hours?

2. Beth, Kelly, Andrea, and Amy are all training for the local marathon.

   a. Beth can run 6.5 miles per hour. At this rate, how far will she run in the first 3 hours of the marathon?

   b. Kelly runs 13.5 miles in 2 hours. What is her rate?
c. Andrea is the slowest runner in the group. She can run 5.5 miles per hour. At this rate, how many miles will she run in the first 3 hours of the marathon?

d. Amy wants to run the 26.2 miles of the marathon in 4.5 hours. At what rate will she have to run to reach this goal?

e. At a workout designed to increase speed, Beth runs 800 meters in 2 1/2 minutes. Kelly runs 1600 meters in 4 1/2 minutes. Who ran the fastest in this workout?

3. Maya left her notebook on the bus and her friend Ariana picked it up for her. On Saturday, they decide to meet to give Maya her notebook. They live 7.5 miles from each other and plan to walk and meet between their homes. Ariana can walk 3 miles per hour; Maya can walk 4.5 miles per hour. Maya makes the suggestion, “It will take the same amount of time if you stay put, and I run 7.5 miles per hour.” Is Maya’s suggestion correct?

Unit rates are also useful when calculating the price of multiple items.

4. Complete each table.

   a. Carpet is sold by the square yard. Classroom carpet sells for $10.50 per square yard.

<table>
<thead>
<tr>
<th>1 yd²</th>
<th>40 yd²</th>
<th>50 yd²</th>
<th>100 yd²</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10.50</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b. Pink Lady apples are sold by the pound. One pound of Pink Lady apples costs $2.99.

<table>
<thead>
<tr>
<th>1 lb</th>
<th>2 lbs</th>
<th>5 lbs</th>
<th>10 lbs</th>
<th>20 lbs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2.99</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
c. Purchases in your county have a 7 percent sales tax added for every dollar of the purchase price.

<table>
<thead>
<tr>
<th></th>
<th>$1</th>
<th>$5</th>
<th>$10</th>
<th>$20</th>
<th>$50</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$0.07</td>
</tr>
</tbody>
</table>

5. How did you use a unit rate to complete each table in Question 4?

TALK the TALK

Shopping for Cereal

Tim and Dan love cereal, but don’t want to spend a lot of money. After scanning the aisle in the grocery store for the lowest prices, the boys make the following statements.

- Tim says, “I found Sweetie Oat Puffs for $0.14 per ounce. That’s the cheapest cereal in the aisle!”
- Dan replies, “It’s not cheaper than Sugar Hoops! The unit price for that is 6.25 oz per dollar.”

Who is correct? Explain your reasoning.
Assignment

Write
Define the term unit rate in your own words.

Remember
Unit rates that involve money, like $1.25 per pound, or speed, like 60 miles per hour, are very common. But not all unit rates are about money or speed.

Practice
1. Write a unit rate for each situation.
   a. 254 words typed in 4 minutes.
   b. 5 trays with 90 ice cubes.
   c. 4 hot dogs eaten in 45 seconds.
   d. 8 hours to drive 528 miles.

2. Shawna needs to buy apples to bake pies for the fair. She needs 13 pounds of apples. At one market, she finds apples selling for $1.89 a pound. At another market she finds a 15-pound bag of apples for $26.99. Which market has the better deal?

3. Dylan needs to buy new contact lenses. His ophthalmologist sells 8-lens boxes in packs of 2 for $52 and 10-lens boxes in packs of 4 for $120. Which option is the better deal?

4. Pets R Us claims in their advertisement that they have the best price in town for ChowChow dog food. They sell 20-pound bags for $16.95. Stuff4Pets also claims to have the best price in town for ChowChow dog food. They are selling 30-pound bags for $24.95. Which store has a valid claim?

5. During his last race, Bryce biked 43 kilometers in 2 hours. If he maintains that same speed, how far will he travel in 3 hours?

Stretch
Describe how sales tax can be a rate. Determine the sales tax for your state or a nearby state and calculate the costs of different items after applying the sales tax.
Review

1. Determine each conversion.
   a. 24 in. = ____ cm
   b. 6 qt = ____ c
   c. 18 ft = ____ m
   d. 5 mi = ____ km
   e. 2.5 m = ____ in.

2. At Union Middle School, 99 boys, or 36% of the boys, play basketball. How many boys attend Union Middle School? Show your work.

3. At Union Middle School, there are a total of 250 girls, 22% of whom play basketball. How many girls at Union Middle School play basketball? Show your work.

4. Determine each value.
   a. 7% of 26
   b. 28% of 90
LESSON 3: Seeing Things Differently • M2-199

You know about special ratios called rates and have used unit rates to convert measurements, determine the better buy, and solve problems about constant speeds. How can you use graphs of rates to solve other types of problems?

LEARNING GOALS
• Represent and identify unit rates using tables and graphs.
• Recognize that \((x, 1)\) and \((1, y)\) are both points on the graph of a unit rate.
• Graph unit rates in real-world situations involving unit pricing and constant speed.
• Compare unit rates based on their graphs.

WARM UP
For each graph, determine if it represents equivalent ratios. Explain your reasoning.

1. \[\text{Graph 1}\]
2. \[\text{Graph 2}\]
3. \[\text{Graph 3}\]
4. \[\text{Graph 4}\]
The Need . . . for Speed

In cars, a speedometer shows the driver the rate at which the car is moving—its speed. Many speedometers are like double number lines arranged in a circular shape. The top number line on this speedometer shows the rate in miles per hour (mph), and the bottom number line shows the rate in kilometers per hour (km/h).

Use the speedometer to estimate.

1. At about what rate, in kilometers per hour, is the car moving if it is traveling at 60 miles per hour?

2. At about what rate, in miles per hour, is the car moving if it is traveling at 60 kilometers per hour?

3. About how long would it take to drive 90 kilometers at 55 miles per hour? Explain your reasoning.
The 6th grade chorus made and sold their own mixture of trail mix at basketball games to raise money for an upcoming trip. During the first basketball game, they sold 1 lb bags for $2.80. They got many requests to sell different sized bags of their trail mix. The group decided to vary the size of the bags, but wanted to make sure that the cost-to-pounds rate stayed the same.

1. Complete the table to display the cost for various quantities of trail mix. Create a graph from your table of values. Be sure to label the axes and name the graph.

<table>
<thead>
<tr>
<th>Trail Mix Weight (lb)</th>
<th>Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
</tr>
</tbody>
</table>

In a unit rate, one or both of the values are 1.

2. Identify two points on the graph that represent unit rates. Write each unit rate in words and explain its meaning.

3. Explain how your graph displays equivalent rates.
A rhombus is considered a Golden Rhombus when the diagonals are in a very specific ratio, known as $\phi$ or phi (pronounced “fi” or “fee”). A Golden Rhombus is shown and your task is to determine the ratio of the diagonals.

A Golden Rhombus

A diagonal is a line segment that connects opposite vertices of a polygon.

ACTIVITY

3.2

Unit Rates and Dimensions

Look around your classroom to identify tools besides a ruler that you can use to measure the lengths of the diagonals.
1. Use standard and non-standard tools to measure the lengths of the diagonals using 6 different units of measure and record them in the table. Be sure to include inches and centimeters as two of your units.

<table>
<thead>
<tr>
<th>Unit of Measure</th>
<th>Length of Diagonal GO</th>
<th>Length of Diagonal RM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Graph the lengths of the diagonals on the coordinate plane.

3. Use a ruler to connect the ratios plotted on the graph. Describe the pattern that the points appear to follow.

4. What does the pattern of ratios on the graph tell us about the ratios?
5. Write two unit rates that relate the length of diagonal $\overline{GO}$ and the length of diagonal $\overline{RM}$.

6. Describe where you can locate the unit rates on the graph.

7. Suppose you measure the Golden Rhombus in units called "ujeni". Use the unit rates to answer each question.
   
   a. If the length of diagonal $\overline{GO}$ is 15 ujeni, what is the length of diagonal $\overline{RM}$ in ujeni?

   b. If the length of diagonal $\overline{RM}$ is 15 ujeni, what is the length of diagonal $\overline{GO}$ in ujeni?
Opened in 1887 and designed to move coal workers from their homes atop Mt. Washington down to the coal factories along the river in Pittsburgh, Pennsylvania, the Duquesne Incline still serves as a mode of transportation for commuters who live in the area.

Jasmine takes the incline to work each morning. The incline is 800 feet long, and it takes 90 seconds to ride from the top of Mt. Washington to the bottom.

1. Identify which of the following statements are true. Explain your reasoning for each.
   
   a. Jasmine travels approximately 178 feet every 20 seconds.
   
   b. She travels approximately 600 feet per minute.
   
   c. In 75 seconds, Jasmine travels approximately 750 feet.
   
   d. She travels approximately 44 feet every 5 seconds.
   
   e. She travels 8.9 feet per second.

2. Plot the correct ratios from Question 1 on the coordinate plane. How can you use the graph to verify correct and incorrect statements from Question 1?
TALK the TALK

Once Upon a Unit Rate

Write a story with unit rates that corresponds to this graph. Include 3 questions and their answers that can be solved using the graph. Be prepared to share your story with the rest of your class.
Assignment

Write
On a graph of equivalent rates explain what each described point represents.
- the point with an x-coordinate of 1
- the point with a y-coordinate of 1

Remember
You can represent rates and unit rates in a variety of different ways—in tables, on graphs, and in stories and other situations.

Practice
Graph the rates in each pair on a coordinate plane. Explain whether or not the rates are equivalent.

1. \( \frac{48 \text{ oz}}{3 \text{ lb}}, \frac{64 \text{ oz}}{4 \text{ lb}} \)
2. \( \frac{150 \text{ mi}}{2.5 \text{ hr}}, \frac{525 \text{ mi}}{8.75 \text{ hr}} \)
3. \( \frac{4.50}{3}, \frac{7.50}{6} \)
4. \( \frac{10}{7}, \frac{12}{8.40} \)
5. \( \frac{200 \text{ cm}}{2 \text{ m}}, \frac{4 \text{ m}}{400 \text{ cm}} \)
6. \( \frac{90 \text{ km}}{1 \text{ hr}}, \frac{180 \text{ km}}{2 \text{ hr}} \)

Stretch
Acceleration is a rate that compares speed with time. Gravity, for example, is acceleration at 9.8 meters per second per second, or \( \frac{9.8 \text{ m}}{1 \text{ s}}, \frac{1 \text{ s}}{1 \text{ s}} \). When an object is in free fall, its speed at any moment is caused by acceleration due to gravity. How fast, in miles per hour, is a body in free fall moving after 4 seconds?
Review

1. A banquet hall is preparing for a wedding with 312 guests. If one table will seat 8 guests, how many tables will be needed for the wedding?

2. Lynn is traveling in Mexico. She exchanges $200 for pesos. If the exchange rate is 19.29 pesos per US dollar, how many pesos should she expect to receive from the exchange?

3. Use benchmark percents to calculate each value. Show your work.
   a. 35% of 142
   b. 22% of 864

4. One popular item at the school store is scented pencils. The pencils come in packs of 24 from the retailer. Write an algebraic expression that represents the total number of pencils the store has available to sell.

5. Determine each sum.
   a. 4.0842 + 13.87 + 6.371
   b. 12.89 + 7.45 + 3.005
There are many situations in which you need to convert measurements to different units. To convert a measurement means to change it to an equivalent measurement in different units. When you convert a measurement to a different unit, the size of the object does not change; only the units and the number of those units change.

Conversions can be written using ratio language. They can also be written symbolically in terms of equality.

<table>
<thead>
<tr>
<th>Ratio Language</th>
<th>Symbolically</th>
</tr>
</thead>
<tbody>
<tr>
<td>For every inch, there are approximately 2.5 centimeters.</td>
<td>1 in. ≈ 2.5 cm</td>
</tr>
<tr>
<td>For every meter, there is approximately 1 yard.</td>
<td>1 m ≈ 1 yd</td>
</tr>
<tr>
<td>For every foot, there are approximately 30 centimeters.</td>
<td>1 ft ≈ 30 cm</td>
</tr>
<tr>
<td>For every 12 inches, there is exactly 1 foot.</td>
<td>12 in. = 1 ft</td>
</tr>
<tr>
<td>For every 1 kilometer, there are exactly 1000 meters.</td>
<td>1 km = 1000 m</td>
</tr>
</tbody>
</table>

A conversion ratio is also called a conversion rate because two quantities that are measured in different units are being compared. For example, you can write the ratio of inches to feet in fractional form: \( \frac{12 \text{ in.}}{1 \text{ ft}} \).

Because these measurement conversions are ratios, you can use ratio reasoning to convert between units, such as double number lines.
For example, the double number line shown represents the ratio of tablespoons to cups.

Using the double number line, you can determine that there are 12 tablespoons in \(\frac{3}{4}\) cup or that there are \(1\frac{1}{2}\) cups in 24 tablespoons.

Using a ratio table is another strategy for converting units. For example, this table represents the ratio of pounds to ounces.

<table>
<thead>
<tr>
<th>Pounds</th>
<th>1</th>
<th>2</th>
<th>(\frac{1}{4})</th>
<th>(\frac{1}{2})</th>
<th>(\frac{3}{8})</th>
<th>(2\frac{1}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ounces</td>
<td>16</td>
<td>32</td>
<td>4</td>
<td>20</td>
<td>8</td>
<td>6</td>
</tr>
</tbody>
</table>

You can add values in different columns to determine new equivalent rates.

Scaling up or down is a similar strategy for determining equivalent ratios that can more easily be used to convert from one unit of measurement to another.

For example, you can use scaling up to determine how many kilograms are in 2.5 pounds. Because you want to determine the number of kilograms for a specific number of pounds, use the conversion rate \(1 \text{ lb} = 0.45 \text{ kg}\) or \(\frac{1 \text{ lb}}{0.45 \text{ kg}}\).

You can also use unit analysis to determine the quantity in pounds that is equivalent to 4.5 kilograms. In unit analysis, you multiply by a form of 1 to rewrite the given measurement in a different unit.
A rate is a ratio in which the two quantities being compared are measured in different units. A **unit rate** is a comparison of two measurements in which the denominator has a value of one unit.

One way to compare the values of products is to calculate the unit rate for each item.

For example, a brand of laundry detergent comes in two different sizes: 26 fluid ounces for $9.75 or 20.5 fluid ounces for $7.25.

The larger bottle of detergent is about 25 fluid ounces for about $10.

\[
\frac{1 \text{ fl oz}}{10} \quad \frac{1 \text{ fl oz}}{25} \quad \frac{1 \text{ fl oz}}{5} \quad \frac{5 \text{ fl oz}}{1} \quad \frac{1 \text{ fl oz}}{0.40}
\]

So, each fluid ounce costs about $0.40, or about $2 per 5 fluid ounces.

The smaller bottle of detergent is about 21 fluid ounces for about $7.

\[
\frac{1 \text{ fl oz}}{7} \quad \frac{1 \text{ fl oz}}{21} \quad \frac{1 \text{ fl oz}}{3} \quad \frac{3 \text{ fl oz}}{1} \quad \frac{1 \text{ fl oz}}{0.33}
\]

So, each fluid ounce costs about $0.33, or about $1 per 3 fluid ounces.

That means that you pay less for each fluid ounce of the smaller bottle of detergent, so it is the better buy.

Unit rates can be written with either quantity as the unit. In the example above, the unit rate was determined as the price per fluid ounce. It can also be written as the number of fluid ounces per dollar. For the larger bottle of detergent, you get about $2.5 per $1, and for the smaller bottle of detergent you get about $3 per $1.
Unit rates are helpful when solving problems about constant speeds.

For example, suppose Sara can ride 50 miles in 4 hours. At this rate, how far will she ride in 7 hours?

\[
\frac{50 \text{ miles}}{4 \text{ hours}} = \frac{12.5 \text{ miles}}{1 \text{ hour}} \quad \text{and} \quad \frac{12.5 \text{ miles}}{1 \text{ hour}} = \frac{87.5 \text{ miles}}{7 \text{ hours}}
\]

Scale down to determine the unit rate. Then scale up to determine the equivalent rate needed to solve the problem.

You can represent rates and unit rates in a variety of different ways—in tables, on graphs, and in stories and other situations.

For example, the 6th grade chorus is selling bags of trail mix in various sizes to raise money for an upcoming trip. The group wants the ratio of cost-to-pounds to stay the same no matter the size of the bag. They decide to sell 1 lb bags for $3.20.

The table shown displays the cost for various quantities of trail mix. These ratios are plotted on the graph and connected with a line.
The lessons in this module build on your knowledge of numeric expressions, patterns, and operations, which you developed throughout elementary school. You will use properties of arithmetic and apply them to algebraic expressions. You will investigate equations and graphs and develop strategies to make sense of and reason about unknown quantities in real-world and mathematical problems.

**Topic 1  Expressions** ................................................................. M3-3
**Topic 2  Equations** ................................................................. M3-83
**Topic 3  Graphing Quantitative Relationships** ............. M3-151
TOPIC 1

Expressions

Emojis in emails and chat messages show different expressions. Mathematical expressions are a little different. But you probably already knew that. 😊

Lesson 1
Relationships Matter
Evaluating Numeric Expressions ................................................. M3-7

Lesson 2
Into the Unknown
Introduction to Algebraic Expressions ................................. M3-23

Lesson 3
Second Verse, Same as the First
Equivalent Expressions .......................................................... M3-35

Lesson 4
Are They Saying the Same Thing?
Verifying Equivalent Expressions ........................................ M3-53

Lesson 5
DVDs and Songs
Using Algebraic Expressions to Analyze and Solve Problems ................................. M3-67
TOPIC 1: EXPRESSIONS
In this topic, students develop their understanding of variables and algebraic expressions. They also formalize their knowledge of powers and evaluate expressions involving whole number exponents, expanding their application of the Order of Operations to include exponents. Students compose algebraic expressions from verbal statements, decompose expressions into their component terms, and evaluate algebraic expressions for given values of the variable. They use algebra tiles and properties of arithmetic and algebra to form equivalent expressions, just as they did in previous lessons with numeric expressions. Students also use tables and graphs to determine if expressions are equivalent, and they write algebraic expressions to model and solve real-world and mathematical problems.

Where have we been?
Students enter grade 6 with knowledge of factors and properties of numbers. They have used the Commutative and Associative Properties in first and third grades and the Order of Operations, although formal terminology may not have been used. These properties, along with the Distributive Property, were reviewed in previous lessons in this course. During elementary school, students wrote expressions with whole number exponents for powers of ten, and they wrote numeric expressions to record verbal descriptions of calculations.

Where are we going?
This topic provides the foundation for future work with algebraic structures, including algebraic equations and inequalities and their representations. Expressions are the foundation of equations. Expertise in writing expressions enables students to write and solve equations for many real-world and mathematical problems. As students continue in the course, they must be able to evaluate expressions and determine whether expressions are equivalent.

Using Algebra Tiles to Model Expressions
Algebra tiles are used to model expressions with variables. For example, this model could show the combination of the expressions $x + 1$ and $2x + 1$. The sum can be written, even when the value of $x$ is not known. The model shows that the sum is $3x + 2$. 
Myth: “I learn best when the instruction matches my learning style.”

If asked, some people will tell you they have a learning style – the expressed preference in learning by seeing images, hearing speech, seeing words, or being able to physically interact with the material. Some people even believe that it is the teacher’s job to present the information in accordance with that preference.

However, it turns out that the best scientific evidence available does not support learning styles. In other words, when an auditory learner receives instruction about content through a visual model, they do just as well as auditory learners who receive spoken information. Students may have a preference for visuals or writing or sound, but sticking to their preference doesn’t help them learn any better. Far more important is ensuring the student is engaged in an interactive learning activity and the new information connects to the student’s prior knowledge.

#mathmythbusted

Talking Points

You can support your student’s learning by resisting the urge, as long as possible, to get to the answer in a problem that your student is working on. Students will learn the algebraic shortcuts that you may know about, but only once they have experience in mathematical reasoning. This may seem to take too long at first. But if you practice asking good questions instead of helping your student arrive at the answer, they will learn to rely on their own knowledge, reasoning, patience, and endurance when struggling with math.

Key Terms

Order of Operations
Evaluate expressions inside parentheses, then exponents, then multiply and divide from left to right, then add and subtract from left to right.

variable
A variable is a symbol, often a letter, that represents a quantity that varies.

algebraic expression
An algebraic expression is a mathematical phrase involving at least one variable, and sometimes numbers and operation symbols.

coefficient
A coefficient is the number that is multiplied by a variable in an algebraic expression.
LESSON 1: Relationships Matter

Evaluating Numeric Expressions

WARM UP
Write each power of ten as a product of factors. Then calculate the product.

1. \(10^2 = \)__ = __
2. \(10^5 = \)__ = __
3. \(10^3 = \)__ = __
4. \(10^4 = \)__ = __
5. \(10^7 = \)__ = __

LEARNING GOALS
- Interpret a number raised to a positive integer power as a repeated product.
- Identify perfect square numbers and perfect cube numbers.
- Write and evaluate numeric expressions involving whole-number exponents.
- Model numeric expressions with two- and three-dimensional figures.
- Evaluate numeric expressions using the Order of Operations.

KEY TERMS
- power
- base
- exponent
- perfect square
- perfect cube
- evaluate a numeric expression
- Order of Operations

You have written and evaluated expressions equivalent to given numbers. Besides the four operations—addition, subtraction, multiplication, and division—are there other structures that can be used in numeric expressions?
Expression Challenge

Recall that an expression in mathematics is a number or a combination of numbers and operations. The number 8 is an expression, and $2 \times 2 + 4$ is also an expression. Both of these expressions are equal to 8.

1. Write an expression that is equal to 10 using only four 2s and any number of math symbols.

2. Write an expression that is equal to 8 using only four 3s and any number of math symbols.

3. Write an expression that is equal to 20 using only one 2 and two 4s and any number of math symbols.
Just as repeated addition can be represented as a multiplication problem, repeated multiplication can be represented as a power. A power has two elements: the base and the exponent.

\[ 2 \times 2 \times 2 \times 2 = 2^4 \]

The base of a power is the factor that is multiplied repeatedly in the power, and the exponent of the power is the number of times the base is used as a factor.

1. Identify the base and exponent in each power. Then, write each power in words.

   a. 7^5  
   b. 4^8

Remember that the area of a rectangle is calculated by multiplying its length by its width. Because all sides of a square have the same length, the area of a square, \( A \), is calculated by multiplying the length of the side, \( s \), by itself. The formula for the area of a square, \( A = s \times s \), can be written as \( A = s^2 \).

In the same way, to calculate the square of a number, you multiply the number by itself.

2. Write the area of each square as a repeated product, as a square number, and as an area in square units.

   a.  
   b. 2.75 m

You can read a power in different ways:
   “2 to the fourth power”
   “2 raised to the fourth power”

In the power \( s^2 \), the base is the side length, \( s \), and the exponent is 2.
Some of the areas that you wrote in Question 1 are called **perfect squares** because they are squares of an integer. For example, 9 is a perfect square because $3 \times 3 = 9$. Another way you can write this mathematical sentence is $3^2 = 9$.

Recall that the volume of a cube is calculated by multiplying its length by its width and its height. Since the length, width, and height of a cube are all the same, the formula for the volume, $V$, of a cube can be written as $V = s \times s \times s$, or $V = s^3$.

In the same way, to calculate the cube of a number, you use the number as a factor three times.

### 3. Write the volume of each cube as a repeated product, as the cube of a number, and as a volume in cubic units.

**a.**

- **2 cm**
- **2 cm**
- **2 cm**

**b.**

- **4 in.**
- **4 in.**
- **4 in.**

**c.**

- **3 ft**
- **3 ft**
- **3 ft**

**d.**

- **5 mm**
- **5 mm**
- **5 mm**

A **perfect cube** is the cube of an integer. For example, 216 is a perfect cube because 6 is a whole number and $6 \times 6 \times 6 = 216$. 

In the power $s^3$, the base is the side length, $s$, and the exponent is 3.

You can read $3^2$ as "3 squared."

You can read $6^3$ as "6 cubed."
Previously, you may have thought about expressions as recipes. For example, the expression $2 + 2$ might have meant “start with 2 and add 2 more.” But as a relationship, $2 + 2$ means “2 combined with 2.”

The Expression Cards at the end of this lesson contain a variety of numeric expressions and models that represent numeric expressions. Cut out the Expression Cards.

1. Consider the different structures of the expressions and the models.
   a. Sort the models in a mathematically meaningful way.
   b. Sort the expressions in a mathematically meaningful way.
   c. Explain how you sorted the Expression Cards.

2. Match the numeric expressions with the models. Select two pairs of cards and explain why each expression matches the model.

Now it’s your turn!

3. Think of a numeric expression. Draw a model to represent that expression. Trade your model with a classmate and write the numeric expression that represents their model. When you both have written your answers, trade back and check your work!
The diagram can be used to determine perfect squares. Daniel drew on the diagram to show that the expression \((4 + 4)^2\) is equivalent to \(8^2\).

1. Explain why \((4 + 4)^2\) is equivalent to \(8^2\) and not equivalent to \(4^2 + 4^2\). Then use the diagram to write other expressions that are equivalent to \(8^2\).
2. Write an equivalent numeric expression for each perfect square.
   a. $6^2$
   b. $12^2$

To **evaluate a numeric expression** means to simplify the expression to a single numeric value.

3. Use the diagram to rewrite the expression $(7 - 3)^2 + (10 - 7)^2$ with fewer terms. Explain your work.

4. Use the diagram to write four numeric expressions. Then explain how to evaluate each expression.

The table shows the cubes of the first 10 whole numbers.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>125</td>
</tr>
<tr>
<td>6</td>
<td>216</td>
</tr>
<tr>
<td>7</td>
<td>343</td>
</tr>
<tr>
<td>8</td>
<td>512</td>
</tr>
<tr>
<td>9</td>
<td>729</td>
</tr>
<tr>
<td>10</td>
<td>1000</td>
</tr>
</tbody>
</table>

5. Write two more equivalent expressions for each. Show how to evaluate the expressions.
   a. $5^3$
   b. $2^3$
Consider the numeric expression $2 \cdot 5^2$.

1. Shae drew a model to represent the expression. Explain how Shae's model represents the expression. Then evaluate the expression.

2. Doug and Miguel each evaluated the expression differently.

Miguel

$2 \cdot 5^2$

$5^2 = 25$

$2 \cdot 25 = 50$

Doug

$2 \cdot 5^2$

$2 \cdot 5 = 10$

$10^2 = 100$

a. What does Miguel's solution tell you about how to evaluate a numeric expression with both multiplication and exponents?

b. Draw a model to represent Doug's solution. Explain how the model is different from Shae's.

Parentheses are symbols used to group numbers and operations. You can think about expressions inside parentheses as a single value.

3. This model represents the expression $(6 + 4) \cdot 3$.

a. Evaluate the expression represented by the model.
b. Draw a model that would represent the expression
$6 + (4 \cdot 3)$ and evaluate the expression.

c. Compare the models and the expressions. How does moving the parentheses change how you draw the model and how you evaluate the expression?

4. Consider the numeric expression $(5 + 3)^2$.

   a. Draw a model to represent this expression.

   b. The numeric expression was evaluated in two different ways, resulting in two different values. Determine which solution is correct. Explain why one solution is correct and state the error that was made in the other solution.

   Solution A
   \[
   (5 + 3)^2
   = 8^2
   = 64
   \]

   Solution B
   \[
   (5 + 3)^2
   = 25 + 9
   = 34
   \]

5. Consider the numeric expression $3 \cdot (7 - 2)$.

   a. Draw a model to represent this expression.

   b. The numeric expression was evaluated in two different ways, resulting in two different values. Determine which solution is correct. Explain why one solution is correct and state the error that was made in the other solution. Cross out the incorrect solution.

   Solution A
   \[
   3 \cdot (7 - 2)
   = 21 - 2
   = 19
   \]

   Solution B
   \[
   3 \cdot (7 - 2)
   = 3(5)
   = 15
   \]
6. A band is playing at a local restaurant for a total of 8 Fridays and will be paid after their last performance. The band advertises their 8 appearances in the local newspaper for a total cost of $400. If the band makes $500 for each appearance, which numeric expression correctly shows the amount of money each of the four members will earn? Explain your reasoning.

<table>
<thead>
<tr>
<th>Expression A</th>
<th>Expression B</th>
</tr>
</thead>
<tbody>
<tr>
<td>((8 \cdot 500 - 400) \div 4)</td>
<td>(8 \cdot 500 - 400 \div 4)</td>
</tr>
</tbody>
</table>

ACTIVITY 1.5

The Order of Operations

There is an Order of Operations, an order in which operations are performed when evaluating any numeric expression. The Order of Operations is a set of rules that ensures the same result every time an expression is evaluated.

Order of Operations Rules

1. Evaluate expressions inside parentheses or grouping symbols.
2. Evaluate exponents.
3. Multiply and divide from left to right.
4. Add and subtract from left to right.

Keep in mind that multiplication and division are of equal importance and evaluated in order from left to right. The same is true for addition and subtraction.
Evaluate each expression using the Order of Operations.

1. \(28 \div 2^2 - 36 \div 3^2\)

2. \(12 + (25 \div 5)^2\)

3. \((12^2 - 48) \times 2\)

4. \(168 \div 2^3 + 3^3 - 20\)

5. \(10 \div (5 - 3) + 2^3\)
TALK the TALK

Order of Operations

Determine whether or not each expression was evaluated correctly. Show the correct work for any incorrect answers.

1. \( \frac{18}{2} \cdot 3^2 \)  
   \( \frac{18}{2} \cdot 9 \)  
   \( \frac{18}{18} \)  
   \( 1 \)

2. \( (15 + 10 ÷ 5) + 8 \)  
   \( (15 + 2) + 8 \)  
   \( 17 + 8 \)  
   \( 25 \)

3. \( 60 - (10 - 6 + 1)^2 \cdot 2 \)  
   \( 60 - (10 - 7)^2 \cdot 2 \)  
   \( 60 - (3)^2 \cdot 2 \)  
   \( 60 - 9 \cdot 2 \)  
   \( 60 - 18 \)  
   \( 42 \)

Each numeric expression has been evaluated correctly and incorrectly. For those that have been evaluated correctly, state how the Order of Operations was used to evaluate the expression. For those expressions that have been evaluated incorrectly, determine the error that was made.

4. \( 2(10 - 1) - 3 \cdot 2 \)  
   \( 2(9) - 3 \cdot 2 \)  
   \( 18 - 3 \cdot 2 \)  
   \( 15 \cdot 2 \)  
   \( 30 \)

5. \( 4 + 3^2 \)  
   \( 4 + 9 \)  
   \( 13 \)

6. \( (2 + 6)^2 \)  
   \( 8^2 \)  
   \( 64 \)
Expression Cards

<table>
<thead>
<tr>
<th>Expression Cards</th>
<th>3 × 2</th>
<th>(3 × 2)^2</th>
<th>(3 + 2)^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 × 2</td>
<td><img src="image1" alt="3 × 2" /></td>
<td>![3 × 2]^2](image2)</td>
<td>![3 + 2]^3](image3)</td>
</tr>
<tr>
<td>3 + 2^2</td>
<td><img src="image4" alt="3 + 2^2" /></td>
<td><img src="image5" alt="3 + 2^2" /></td>
<td><img src="image6" alt="3 + 2^2" /></td>
</tr>
<tr>
<td>3^3 + 2^3</td>
<td><img src="image7" alt="3^3 + 2^3" /></td>
<td><img src="image8" alt="3^3 + 2^3" /></td>
<td><img src="image9" alt="3^3 + 2^3" /></td>
</tr>
<tr>
<td>3 × 2 + 3</td>
<td><img src="image10" alt="3 × 2 + 3" /></td>
<td><img src="image11" alt="3 × 2 + 3" /></td>
<td><img src="image12" alt="3 × 2 + 3" /></td>
</tr>
<tr>
<td>3^2 + 2^2</td>
<td><img src="image13" alt="3^2 + 2^2" /></td>
<td><img src="image14" alt="3^2 + 2^2" /></td>
<td><img src="image15" alt="3^2 + 2^2" /></td>
</tr>
<tr>
<td>2^3</td>
<td><img src="image16" alt="2^3" /></td>
<td><img src="image17" alt="2^3" /></td>
<td><img src="image18" alt="2^3" /></td>
</tr>
</tbody>
</table>
Assignment

Write
Write your own mnemonic for the Order of Operations.

Remember
Memorize the first 15 squares and 10 cubes.

<table>
<thead>
<tr>
<th>Perfect Squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^2 = 1$</td>
</tr>
<tr>
<td>$6^2 = 36$</td>
</tr>
<tr>
<td>$11^2 = 121$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Perfect Cubes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^3 = 1$</td>
</tr>
<tr>
<td>$6^3 = 216$</td>
</tr>
</tbody>
</table>

Practice
Use the Order of Operations to evaluate each numeric expression.
1. $4^2 \cdot 3$
2. $3^3 - 14 \div 2 + 5$
3. $17 - 2^3$
4. $144 \div 6^2 \cdot 8 + 2^2$
5. $32 \div 4^2$
6. $2^4 - 3 \cdot 5 + 9$
7. $9 + 5^2 - 2 \cdot 3^2$
8. $11^2 - 7 \cdot 6 - 4^3 \div 2$

Stretch
Evaluate each power raised to a power.
1. $(3^?)^2$
2. $(5^?)^4$
3. $(4^?)^2$
Review

Graph each rate in the given pair on a coordinate plane. Explain whether or not the rates are equivalent.

1. \[
\frac{15 \text{ cups flour}}{8.25 \text{ cups sugar}} \quad \frac{5 \text{ cups flour}}{2.75 \text{ cups sugar}}
\]

2. \[
\frac{245 \text{ mi}}{3.5 \text{ h}} \quad \frac{150 \text{ mi}}{2 \text{ h}}
\]

Calculate each conversion.

3. 4 grams = _____ milligrams
4. 6400 ounces = _____ pounds

Determine each sum.

5. \[
\frac{6}{7} + 3\frac{1}{5}
\]

6. \[
1\frac{2}{3} + 4\frac{1}{4}
\]
LEARNING GOALS
• Write algebraic expressions to represent real-world and mathematical situations.
• Match algebraic and verbal expressions.
• Identify parts of an algebraic expression using mathematical terms.
• Evaluate algebraic expressions at specific values of their variables.

KEY TERMS
• variable
• algebraic expression
• coefficient
• term
• evaluate an algebraic expression

You have written and evaluated expressions made up of numbers, but often expressions are made up of numbers and letters. What situations can be represented by expressions with letters and how do you evaluate them?
Do You Speak Math?

Rewrite each statement using symbols.

1. fourteen more than six

2. six more than fourteen

3. seven less than thirteen

4. thirteen less than seven

5. twenty-three subtracted from thirty

6. thirty subtracted from twenty-three

7. the quotient of twelve divided by four

8. the quotient of four divided by twelve

9. one-fourth of twenty-eight

10. two to the seventh power

11. seven squared
Consider the quantity that changes as you think about the situations in Question 1.

1. A school lunch costs $1.85 for each student. For each situation, write a numeric expression to determine how much money is collected. Then evaluate the expression.
   
   a. Fifty-five students purchase a school lunch.
   
   b. One hundred twenty-six students purchase a school lunch.
   
   c. Two hundred thirteen students purchase a school lunch.
   
   d. One thousand five hundred twelve students purchase a school lunch.

2. Write a sentence to describe how you can determine the amount of money collected for any number of students buying school lunches.

In Question 1 there is one quantity that changes or varies—the number of students who bought school lunches. In mathematics, letters are often used to represent quantities that vary. These letters are called variables, and they help you write algebraic expressions to represent situations. An algebraic expression is an expression that has at least one variable.

3. Write an algebraic expression to represent the total amount of money collected for any number of students buying school lunches.
A number that is multiplied by a variable in an algebraic expression is called a coefficient.

4. Identify a coefficient in the expression you wrote in Question 3.

5. The cost to rent a skating rink is $215 for a two-hour party. The cost will be shared equally among all the people who attend the party. For each number of attendees, write a numeric expression to determine how much each person will pay. Then evaluate the expression.
   a. 25 attendees
   b. 81 attendees
   c. 108 attendees
   d. Write an algebraic expression to represent how much each person will pay to attend the skate party.

6. Jimmy has three 300-minute international calling cards.
   a. Complete the table to determine how many minutes are left on each card after each call.

<table>
<thead>
<tr>
<th>Minutes on Card</th>
<th>Duration of Call</th>
<th>Minutes Left on Card</th>
</tr>
</thead>
<tbody>
<tr>
<td>300</td>
<td>33 min</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>57 min</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>1 h 17 min</td>
<td></td>
</tr>
</tbody>
</table>

   b. Write an algebraic expression that represents the number of minutes remaining after each call on each card.
7. Write an algebraic expression to represent each situation. Identify the coefficient(s).

   a. Ben is selling tickets to the school play. How many will he have left if he starts with \( t \) tickets and sells 125 tickets?

   b. A plane descends to \( \frac{5}{6} \) of its cruising altitude, \( a \). What is its new altitude?

   c. A cube has an edge length of \( s \).
     1. What is the volume of the cube?
     2. What is the surface area of the cube?

   d. Used paperback books cost $6.25 each with an additional shipping and handling cost of $8.75. What is the cost of \( x \) books?

   e. Chairs cost $35, and sofas cost $75. How much does it cost to purchase \( x \) chairs and \( y \) sofas?

8. Write an algebraic expression to represent each word expression.

   a. the quotient of a number \( n \) divided by 7

   b. 5 more than \( c \)
Let’s play Expression Explosion! You teacher is going to hand out cards. Your goal is to identify the written or algebraic expression that corresponds to your card.

Record your pair of matching algebraic and written expressions.

1. How can you be sure that you have found the correct match?

- c. $m$ less than 9
- d. one-fourth of a number $n$
- e. fourteen less than three times a number $n$
- f. six times a number $n$ subtracted from 21
- g. one-fourth of a number $n$ minus 6
- h. ten times the square of a number $w$ divided by 12
As you learned previously, an algebraic expression contains at least one variable and sometimes numbers and operations. A term of an algebraic expression is a number, variable, or product of numbers and variables.

**WORKED EXAMPLE**

Consider the expression $3x + 4y - 7$.

The expression has three terms: $3x$, $4y$, and $7$. The operation between the first two terms is addition, and the operation between the second and third term is subtraction.

1. Consider two algebraic expressions: $8 + 5x$ and $8 - 5x$
   
   a. Identify the terms in each algebraic expression.

   b. Identify the operation between each term in each algebraic expression.

   c. What is the same in both expressions?

   d. What is different in the expressions?
2. Identify the number of terms, and then the terms themselves for each algebraic expression.

a. \( 4 - 3x \)

b. \( 4a - 9 + 3a \)

c. \( 7b - 9x + 3a - 12 \)

---

**ACTIVITY 2.4 Evaluating Algebraic Expressions**

To **evaluate an algebraic expression** means to determine the value of the expression for a given value of each variable. When you evaluate an algebraic expression, you substitute the given values for the variables, and then determine the value of the expression.

1. Write a sentence to describe the meaning of each algebraic expression. Then, evaluate the algebraic expression for the given value.

a. \( 3x - 4, \text{ for } x = 10 \)

b. \( 11 - s, \text{ for } s = 2 \)

c. \( 10 - z, \text{ for } z = 8 \)

d. \( 5 - \frac{y}{4}, \text{ for } y = 2 \)

Don’t forget to use the Order of Operations when evaluating an algebraic expression.
e. \(7 + 5a\), for \(a = 20\)

f. \(\frac{b}{4}\), for \(b = 8\)

2. Complete each table.

<table>
<thead>
<tr>
<th>(h)</th>
<th>(3h - 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>(\frac{7}{3})</td>
<td></td>
</tr>
<tr>
<td>5.1</td>
<td></td>
</tr>
<tr>
<td>(\frac{5}{6})</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(m)</th>
<th>(1 + m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(\frac{2}{3})</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1.7</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(z)</th>
<th>(\frac{2z}{3} + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(p)</th>
<th>(0.5p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td></td>
</tr>
</tbody>
</table>
TALK the TALK

Expression Construction

1. Construct an algebraic expression for each description.

   a. There are 2 terms. The first term is a constant. It is added to the second term, which is a product of a number and a variable.

   b. There are 4 terms. The first term is a variable divided by 11. This is added to a second term, which is a constant. The third term is a second variable multiplied by three-fourths. The third term is subtracted from the first 2 terms. The last term, a different constant, is added to the other 3 terms.

   c. The cube of a variable subtracted from a constant and then added to the square of the same variable.

   d. A number multiplied by the square of a variable minus a number multiplied by the same variable minus a constant.

It is your turn to challenge your classmates!

2. Create a description for an algebraic expression and swap descriptions with a classmate. After you receive the algebraic expression back from your classmate, answer Question 3.

3. Did your classmate write an expression that fits your description?
Assignment

Write
Complete each statement with the correct term: algebraic expression, variable, evaluate an algebraic expression, constant, coefficient.
1. A(n) ___________ is a letter used to represent a quantity that varies.
2. A(n) ___________ is a number, or quantity, that a variable is multiplied by in an algebraic expression.
3. A number, or quantity, that does not change its value is called a(n) ___________.
4. A mathematical phrase involving at least one variable is called a(n) ___________.
5. To ______________ means to determine the value of the expression.

Practice
Write an algebraic expression to represent each situation.
1. A T-shirt costs $5.99.
   a. How much will you spend if you buy x T-shirts?
   b. Evaluate your expression to calculate the amount of money you will spend if you buy 4 shirts or 10 shirts.
2. You have 7 folders and you want to put the same number of pages in each folder.
   a. If you have a total of p pages, how many pages will be in each folder?
   b. Evaluate your expression to calculate the number of pages in each folder if you have 147 pages or 245 pages.
3. You have a coupon for $5 off your total bill at Mama’s Meals on Main.
   a. How much will you pay after using the coupon if your bill was b dollars?
   b. Evaluate your expression to calculate the amount you will pay if your bill was $23.45 or $54.83.
4. You have already read two and a half hours for the Read-a-Thon.
   a. How long will you have read if you read an additional h hours?
   b. Evaluate your expression to calculate the amount of time you will have read if you read 3 or $5.5$ additional hours.

Remember
Whenever you perform the same mathematical process over and over, you can write a mathematical phrase, called an algebraic expression, to represent the situation.
Write an algebraic expression that represents each verbal expression.
5. six times a number plus 3
6. four times a number subtracted from 2
7. a number squared divided by 2 and added to 16
8. five plus a number and then multiplied by 8

Identify the number of terms and then the terms themselves for each algebraic expression.
9. 6y + 14
10. 7x – 3y + 12z
11. 104a + 224b

Evaluate each algebraic expression for the given value.
12. 34 – y² for y = 5
13. m³ + 18 for m = 2
14. d/5 + 42 for d = 70

Stretch
Farmer Lyndi raises chickens and goats.
1. Write an expression for the total number of animal legs on Lyndi’s farm.
2. How many animal legs are on the farm if Lyndi has 16 chickens and 6 goats?
3. Suppose Lyndi counted 74 animal legs on the farm. How many of each animal might Lyndi have on the farm?

Review
Evaluate each numeric expression.
1. 56 ÷ 8 + 3 · 6
2. 9 · 8 – 29 + 30 ÷ 15 – 15

Determine which is the better buy.
3. $12.99 for 42 ounces or $2.99 for 10 ounces
4. 3 pounds for $5.00 or $1.50 per pound

Determine at least two equivalent ratios for each given ratio.
5. \[
\frac{2 \text{ eggs}}{5 \text{ cups of milk}}
\]
6. \[
\frac{20 \text{ red}}{12 \text{ blue}}
\]
WARM UP
Evaluate each expression.

1. \(5 \div \frac{3}{4}\)
2. \(0.24 \div 0.6\)
3. \(\frac{(14 + 8)}{2}\)
4. \(\frac{14}{2} + \frac{8}{2}\)

5. What do you notice about the answers to Questions 3 and 4?

LEARNING GOALS
• Model algebraic expressions with algebra tiles.
• Simplify algebraic expressions using algebra tiles.
• Simplify algebraic expressions using the associative, commutative, and distributive properties.
• Apply properties of operations to create equivalent expressions.
• Rewrite expressions as the product of two factors.

KEY TERMS
• like terms
• Distributive Property
• equivalent expressions

You have evaluated numeric expressions and written and evaluated algebraic expressions. How do you combine algebraic expressions, like you did with numeric expressions, into as few terms as possible?
Packing for a Camping Trip

Jaden and Jerome, twin brothers, are packing for a weekend camping trip. They lay out the following items to go in the suitcase.

**Jaden:**
- 3 shirts
- 2 pairs of pants

**Jerome:**
- 4 shirts
- 1 pair of pants
- 2 pairs of shorts
- 1 pair of swim trunks

1. How many shirts and pairs of pants is each brother packing? Together, how many shirts and pairs of pants are they packing?

<table>
<thead>
<tr>
<th>Shirts</th>
<th>Pants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jaden</td>
<td>5</td>
</tr>
<tr>
<td>Jerome</td>
<td>6</td>
</tr>
</tbody>
</table>

Together:
- 9 shirts
- 6 pairs of pants

**Shirts**

<table>
<thead>
<tr>
<th>Jaden</th>
<th>Jerome</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

**Pants**

<table>
<thead>
<tr>
<th>Jaden</th>
<th>Jerome</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Together:

- 3 pairs of pants
- 1 pair of swim trunks
Your teacher has provided you with algebra tiles.

2. How can you use algebra tiles to model the number of shirts packed by each brother and the number of shirts they packed together?

### Activity 3.1

**Algebra Tiles and Combining Like Terms**

As you may have seen in the previous activity, when using algebra tiles to model situations and expressions, it is important to have a shared meaning for each differently-sized algebra tile.

Your teacher will hold up each differently-sized algebra tile and tell you the conventional value of each.

1. Sketch each tile and record its value.

2. Represent each numeric or algebraic expression using algebra tiles. Write an addition expression that highlights the different tiles used in the model. Then, sketch the model below the expression.
   
   a. 3  
   b. 3x  
   c. 3x²
In an algebraic expression, **like terms** are two or more terms that have the same variable raised to the same power. The coefficients of like terms can be different. Let’s start our exploration of combining like terms with a review of the properties of arithmetic and algebra that you will use to combine terms.

The expression you wrote in each part of Question 2 was made up of like terms. All tiles that are the same size and have the same value represent like terms.

3. **Given the algebra tile model, write an addition expression that highlights the different tiles in the model. Then, if necessary, combine like terms and write the expression using as few terms as possible.**

4. **Analyze the last expression you wrote in Question 3.**
   
   a. How many terms are in your expression with the fewest terms? How does this relate to the algebra tile model?
   
   b. What is the greatest exponent in the expression?

   c. What is the coefficient of \(x\) in the expression? How does this relate to your algebra tile model?
5. Consider the model.

\[
\begin{array}{c}
\times \\
\times \\
\times \\
\times \\
\times \\
\times \\
\times \\
\times \\
\times \\
\end{array}
\quad \begin{array}{c}
y^2 \\
y^2 \\
y^2 \\
y^2 \\
y^2 \\
\end{array}
\]

a. Write an addition expression that highlights the different tiles in the model.

b. Rearrange the tiles to combine all of the like tiles. How many terms does your expression have now?

c. Write the new algebraic expression represented.

6. Represent the algebraic expression \(3x^2 + x + 2\) using algebra tiles. How many types of tiles are needed?

Algebra tiles are helpful tools for combining like terms in algebraic expressions. However, because they only represent whole number tiles, they cannot be used to model all algebraic expressions.

7. Use what you have learned about combining like terms to rewrite each algebraic expression with as few terms as possible.

   a. \(2x + 3x - 4.5x\)
   
   b. \(\frac{3}{2}y + 2 + 4y + 1\frac{1}{4}\)
   
   c. \(4.5x + 6y - 3.5x + 7\)
   
   d. \(\frac{3}{4}x + 2 + \frac{3}{8}x\)
   
   e. \(5x + 2y + \frac{1}{3}x^2 - 3x\)

So, combining like terms means to add or subtract terms with the same variables. Like \(3x + 5x\). That’s \(8x\)
Let’s use algebra tiles to explore rewriting algebraic expressions with the Distributive Property.

WORKED EXAMPLE

Consider the expression 5(x + 1). This expression has two factors: 5 and the quantity (x + 1). You can use the Distributive Property to rewrite this expression. In this case, multiply the 5 by each term of the quantity (x + 1). The model using algebra tiles is shown.

\[
\begin{align*}
5(x + 1) &= 5x + 5
\end{align*}
\]
1. Analyze the parts of the mathematical expressions in the worked example. Explain each response.
   
a. Which expression, $5(x + 1)$ or $5x + 5$, shows a product of two factors?
   
   b. How many terms are in $5x + 5$?
   
   c. The number 5 is a coefficient in which expression?
   
2. Create a model of each expression using your algebra tiles. Then, sketch the model and rewrite the expression using the Distributive Property.
   
a. $4(2x + 1)$
   
b. $(3x + 1)^2$
3. Rewrite each expression using the Distributive Property. Then, combine like terms if possible.

a. $2(x + 4)$

b. $\frac{2}{3}(6x + 12)$

c. $2(x + 5) + 4(x + 7)$

d. $5x + 2(3x - 7)$

e. $2(y + 5) + 2(x + 5)$

f. $\frac{1}{2}(4x + 2) + 8x$

So far in this activity, you have multiplied expressions together using the Distributive Property. Now let’s think about how to divide expressions.
How do you think the Distributive Property will play a part in dividing expressions? Let’s find out.

4. Consider the expression $(4x + 8) \div 4$, which can also be rewritten as $\frac{4x + 8}{4}$.
   
   a. First, represent $4x + 8$ using your algebra tiles. Sketch the model you create.

   b. Next, divide your algebra tile model into four equal groups. Then, sketch the model you created with your algebra tiles.

   c. Write an expression to represent each group from your sketches in part (b).

   d. Verify you created equal groups by multiplying your expression from part (c) by 4. The product you calculate should equal $4x + 8$. 

I know that multiplication and division are inverse operations. So, I should start thinking in reverse.
Let’s consider the division expression from Question 4.

**WORKED EXAMPLE**
You can rewrite an expression of the form \( \frac{4x + 8}{4} \) using the Distributive Property.

\[
\frac{4x + 8}{4} = \frac{4x}{4} + \frac{8}{4}
= 1x + 2
= x + 2
\]

So, \( \frac{4x + 8}{4} = x + 2 \)

The model you created in Question 4 is an example that shows that the Distributive Property can be used with division as well as with multiplication.

5. Consider the expression \( \frac{2x + 6y + 4}{2} \).

a. Use algebra tiles to represent the division expression.

b. Rewrite the division expression using the Distributive Property. Then, simplify the expression.

\[
\frac{2x + 6y + 4}{2} = \frac{2x}{2} + \frac{6y}{2} + \frac{4}{2}
\]

c. Verify that your answer is correct.

\[
= \_\_\_\_\_\_\_\_\_\_\短时间内\*
\]

M3-44 • **TOPIC 1:** Expressions
Zachary thinks he can simplify algebraic expressions that use the Distributive Property with division without using algebra tiles. He wants to rewrite \( \frac{6 + 3(x + 1)}{3} \) in as few terms as possible and proposes two different methods.

6. Analyze each correct method.

Method 1
\[
\frac{6 + 3(x + 1)}{3} = \frac{6}{3} + \frac{3(x + 1)}{3} = 2 + (x + 1) = x + 3
\]

Method 2
\[
\frac{6 + 3(x + 1)}{3} = \frac{6}{3} + \frac{3(x + 1)}{3} = 2 + (x + 1) = x + 3
\]

a. Explain the reasoning used in each method.

b. Which method do you prefer. Why?
You have used the Distributive Property to multiply and divide algebraic expressions by a given value. The Distributive Property can also be used to rewrite an algebraic expression as a product of two factors: a constant and a sum of terms.

You can write any expression as a product of two factors. In many types of math problems, you often need the coefficient of a variable to be 1. Let’s explore how to use the Distributive Property — without algebra tiles—to rewrite expressions so that the coefficient of the variable is 1.

1. Consider the expression $3x + 6$.

   a. Identify the coefficient of the variable term.

   b. Use the Distributive Property to rewrite the expression as the product of two factors: the coefficient and a sum of terms.

   c. How can you check your work?
Using the Distributive Property to rewrite the sum of two terms as the product of two factors is also referred to as factoring expressions. In the expression $3x + 6$, you factored out the common factor of 3 from each term and rewrote the expression as $3(x + 2)$. In other words, you divided 3 from each term and wrote the expression as the product of 3 and the sum of the remaining factors, $(x + 2)$.

You can use the same strategy to rewrite an algebraic expression so that the coefficient of the variable is 1 even if the terms do not have common factors.

**WORKED EXAMPLE**

Let’s rewrite the expression $4x - 7$ so the coefficient of the variable is 1.

To rewrite the expression, factor out the coefficient 4 from each term. The equivalent expression is the product of the coefficient and the difference of the remaining factors.

$$4x - 7 = 4\left(\frac{4x}{4} - \frac{7}{4}\right) = 4\left(x - \frac{7}{4}\right)$$

2. Use the Distributive Property to check that the new expression is equivalent to the original expression in the Worked Example.

3. Rewrite each expression as the product of two factors. Check your answers.
   
   a. $4x + 5$  
   b. $8x - 3$

   c. $\frac{1}{2}x - 4$  
   d. $1.1x + 1.21$
Rewrite each expression using the Distributive Property.

1. \( \frac{32 + 4x}{4} \)  
2. \( 15x - 10 \)

3. \( \frac{3(x + 1) + 12}{3} \)  
4. \( 2\frac{1}{2} + \frac{1}{4}x \)

Rewrite each algebraic expression in as few terms as possible.

5. \( 3x + 5y - 3x + 2y \)  
6. \( 4x^2 + 4y + 3x + 2y^2 \)

7. \( 7x + 5 - 6x + 2 \)  
8. \( x^2 + 5y + 4x^2 - 3y \)
Rewrite each algebraic expression by applying the Distributive Property and then combining like terms.

9. \(4(x + 5y) - 3x\)  
   10. \(2(2x + 5y) + 3(x + 3y)\)  

11. \(3x + 5(2x + 7)\)  
   12. \(\frac{4x + 6y}{2} - 3y\)  

13. \(3(x + 2y) + \frac{3x - 9y}{3}\)  
   14. \(2(x + 3y) + 4(x + 5y) - 3x\)
TALK the TALK

Write Right

Mr. Martin asked his class to write expressions equivalent to $7(3a + 5b)$ and $8 + 3(2x + 5)$ and got 5 different responses for each. For each response, determine if the original expression was rewritten correctly. For those not rewritten correctly, describe the mistake that was made in rewriting the expression.

1. $7(3a + 5b)$
   a. $10a + 12b$
   b. $7(3a) + 7(5b)$
   c. $21a + 5b$
   d. $21a + 35b$
   e. $7(8ab)$

2. $8 + 3(2x + 5)$
   a. $8 + 3 \cdot 2x + 3 \cdot 5$
   b. $23 + 6x$
   c. $11(2x + 5)$
   d. $8 + 6x + 15$
   e. $13 + 6x$
Assignment

Write
Describe 3 different ways that you can use the Distributive Property to rewrite expressions. Provide an example for each.

Remember
To rewrite an algebraic expression with as few terms as possible, use the properties of arithmetic and the Order of Operations.

An algebraic expression containing terms can be written as the product of two factors by applying the Distributive Property.

Practice
1. Represent each algebraic expression by sketching algebra tiles. Rewrite the expression in a fewer number of terms, if possible.
   a. \( x^2 + 2y^2 + 5 \)
   b. \( y^2 + 3y + 1 + y \)

2. Rewrite each expression by combining like terms.
   a. \( 4.5x + (6y - 3.5x) + 7 \)
   b. \( \left( \frac{2}{3}y + \frac{5}{8}x + \frac{1}{4} \right) + \left( \frac{1}{4}x + \frac{1}{2} \right) \)

3. Nelson is going on an overnight family reunion camping trip. He is in charge of bringing the wood for the campfire. He will start the fire with 6 logs and then plans to add 3 logs for each hour the fire burns.
   a. Represent the number of logs he will use as an algebraic expression.
   b. Suppose the family decides to stay for 2 nights next year. Write the expression for the number of logs they would need for 2 nights.
   c. Create a model of the situation in part (b) using your algebra tiles, and then sketch the model.
   d. Rewrite the expression in part (c) using as few terms as possible.
   e. Nelson’s cousin believes they will only need one-third of the firewood Nelson brings for one night. Represent this as an expression and then use the Distributive Property to rewrite the expression.
   f. There are several family members who will be visiting for the day only. The campground charges $6 per car, plus $2 per visitor. One of the families brings a coupon for $3 off their total fee. Write the expression that represents their total cost for the day. Define the variables.
   g. The two oldest uncles at the reunion insist on paying the bill for the daily visitors. They will split the bill equally. Represent the amount of money each uncle will pay as an expression. Then use the Distributive Property to rewrite the expression.

4. Rewrite each expression by applying the Distributive Property and combining like terms.
   a. \( 7(2x + y) + 5(x + 4y) \)
   b. \( 9x + 6y + \frac{12y + 16x}{4} \)
   c. \( \frac{6(x + 1) + 30}{6} \)

5. Rewrite each expression as a product of two factors, so that the coefficient of the variable is 1.
   a. \( 6x + 7 \)
   b. \( \frac{2}{3}x + 8 \)

LESSON 3: Second Verse, Same as the First • M3-51
**Stretch**

1. Simplify the algebraic expression to include as few terms as possible.

\[3[2x + 4(5y + 1)] + \frac{1}{4}[8y + 12\left(\frac{2}{3}x + \frac{1}{6}\right)]\]

2. Rewrite each algebraic expression as the product of two factors, such that the coefficient of the term with the highest exponent is 1.
   a. \(2x^2 + 5x + 1\)
   b. \(\frac{3}{4}x^3 - 9x^2 + \frac{2}{3}x + 10\)
   c. \(2.6y^2 + 3.9y - 12.48\)

**Review**

1. Sheldon Elementary School has a school store that sells many items including folders, pencils, erasers, and novelty items. The parent association is in charge of buying items for the store.
   a. One popular item at the store is scented pencils that come in packs of 24 from the retailer. Write an algebraic expression that represents the total number of scented pencils they will have available to sell. Let \(p\) represent the number of packs of scented pencils.
   b. Another popular item at the store is animal-themed folders. Each pack of folders contains 6 folders. The store currently has 4 packs in the store and would like to order more. Write an algebraic expression for the total number of folders they will have after they order more folders. Let \(x\) represent the number of packs of folders they buy.
   c. The latest fad is animal-shaped rubber bracelets. The bracelets come in a pack of 24. Write an algebraic expression that represents the cost of each bracelet. Let \(c\) represent the cost of a pack of 24 bracelets.

2. Determine which rate is faster.
   a. 185 miles in 3 hours or 490 miles in 8 hours
   b. 70 miles per hour or 100 kilometers per hour

3. Calculate the volume of each solid formed by rectangular prisms.
   a. [Diagram of rectangular prism with dimensions: 2 cm x 2 cm x 6 cm]
   b. [Diagram of rectangular prism with dimensions: 2 yd x 2 yd x 4 yd]

M3-52 • **TOPIC 1**: Expressions
LESSON 4: Are They Saying the Same Thing?  •  M3-53

**LEARNING GOALS**
- Compare expressions using properties, tables, and graphs.
- Identify when two expressions are equivalent.
- Determine if two expressions are equivalent.

**WARM UP**
Determine which pairs of ratios are equivalent. Explain how you know.

1. 5:7 and 100:140
2. 42:48 and 14:15
3. 105:100 and 20:21
4. 9:12 and 60:80

You know how to use the Distributive Property and combine like terms to write equivalent expressions. How can you determine if two given expressions are equivalent?
Property Sort

Cut out the Property Cards located at the end of the lesson.

On each card is one representation of a property of numbers or operations that you have used in the past to rewrite and evaluate numeric expressions.

1. Sort the cards according to the property named or illustrated on the cards. Create a table that shows your final sorting.

2. Using complete sentences, write an explanation for how each picture illustrates its property.
Two algebraic expressions are equivalent expressions if, when any values are substituted for the variables, the results are equal.

While it’s not realistic to test each expression for every possible value for the unknown, you can examine the characteristics of each expression in the different representations:

- a table of values
- rewritten expressions using the properties
- a graph of both expressions

Let’s explore each representation.

Consider the two expressions $2(x + 2) + 3x$ and $5x + 4$.

1. Use a table to evaluate each expression for different values of the variable.

   a. Complete the table of values for each value of $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$2(x + 2) + 3x$</th>
<th>$5x + 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b. What can you determine based on the values in the table?

   c. What would you need to know to be able to verify that the two expressions are equivalent?
2. Rewrite the given expression and identify the property applied at each step.

\[2(x + 2) + 3x\]  
\[= 2x + \underline{\hspace{2cm}} + 3x\]  
\[= \underline{\hspace{2cm}} x + 4\]  

Given: \[\text{Combine Like Terms/Addition}\]

3. Are the two expressions equivalent? Explain.

You can also use a graph to determine or verify if two expressions are equivalent.

4. Use the table of values to sketch the graph of both expressions on the coordinate plane.

a. Plot the values for each expression on the coordinate plane. Use a □ to represent the values from the first expression and a △ for the values from the second expression. Then, connect the results for each expression with a line.

b. How does the graph demonstrate that the two expressions are equivalent?
Now, let’s consider the expressions $2x + 5$ and $2(x + 5)$.

5. Use a table to evaluate each expression for different values of the variable.

a. Complete the table of values for each value of $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$2x + 5$</th>
<th>$2(x + 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. What can you determine based on the values in the table?

6. Use the Distributive Property to rewrite the second expression.

7. Are the two expressions equivalent? Explain your reasoning.
8. Use the table of values to sketch the graph of both expressions on the coordinate plane.

a. Plot the values for each expression on the coordinate plane. Use a □ to represent the values from the first expression and a △ for the values from the second expression. Then, connect the results for each expression with a line.

b. What does the graph tell you about the equivalence of the two expressions?

For each pair of expressions, use a table, properties, and a graph to determine if the expressions are equivalent.

9. \((3x + 8) + (6 - x)\) and \(4x + 14\)

a. | x   | \((3x + 8) + (6 - x)\) | \(4x + 14\) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b. \( (3x + 8) + (6 - x) \)

\[
= (3x + 8) + (6 - x) \\
= 3x + (8 + 6) - x \\
= 3x + ____ - x \\
= ____ + 3x - x \\
= ___
\]

**Given**

**Commutative Property of Addition**

**Combine Like Terms**

---

c. 

![Graph](image)

d. Are the two expressions equivalent? Explain using all three representations.
10. $x + 3(2x + 1)$ and $7x + 3$

a. | $x$ | $x + 3(2x + 1)$ | $7x + 3$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. $x + 3(2x + 1)$

Given

$= x +$  

$= \underline{\phantom{0}}$

d. Are the two expressions equivalent? Explain using all three representations.
### TALK the TALK 🌟

#### Property Management

For each step of the simplification of the expression, identify the property or operation applied.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Number Property or Operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $10 \cdot 4x + 3(2x + 1)$</td>
<td>Given</td>
</tr>
<tr>
<td>$= (10 \cdot 4)x + 3(2x + 1)$</td>
<td></td>
</tr>
<tr>
<td>$= 40x + 3(2x + 1)$</td>
<td>Multiplication</td>
</tr>
<tr>
<td>$= 40x + 6x + 3$</td>
<td></td>
</tr>
<tr>
<td>$= 46x + 3$</td>
<td></td>
</tr>
<tr>
<td>2. $20 + (6 + x) + 7$</td>
<td>Given</td>
</tr>
<tr>
<td>$= 20 + (x + 6) + 7$</td>
<td></td>
</tr>
<tr>
<td>$= 20 + x + (6 + 7)$</td>
<td>Associative Property of Addition</td>
</tr>
<tr>
<td>$= 20 + x + 13$</td>
<td></td>
</tr>
<tr>
<td>$= x + 20 + 13$</td>
<td></td>
</tr>
<tr>
<td>$= x + 33$</td>
<td></td>
</tr>
<tr>
<td>3. $7x + \frac{12x - 8}{4} + 5x$</td>
<td>Given</td>
</tr>
<tr>
<td>$= 7x + 3x - 2 + 5x$</td>
<td></td>
</tr>
<tr>
<td>$= 10x - 2 + 5x$</td>
<td></td>
</tr>
<tr>
<td>$= 10x + 5x - 2$</td>
<td></td>
</tr>
<tr>
<td>$= 15x - 2$</td>
<td></td>
</tr>
</tbody>
</table>
4. Rewrite \( \frac{2(x + 5) - 4}{2} - x \) using the fewest terms possible. Justify each step with a property or operation.

5. How can you use another representation to check that your answer in Question 4 is equivalent to \( \frac{2(x + 5) - 4}{2} - x \)?
<table>
<thead>
<tr>
<th>Commutative Property of Addition</th>
<th>Associative Property of Multiplication</th>
<th>$a (b + c) = ab + ac$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Distributive Property</strong></td>
<td><strong>Commutative Property of Multiplication</strong></td>
<td>$(13 \cdot 2) \cdot 5 = 13 \cdot (2 \cdot 5)$</td>
</tr>
<tr>
<td>$6 \cdot 5 = 5 \cdot 6$</td>
<td><strong>Associative Property of Addition</strong></td>
<td>$5 (10 + 2) = 5 \cdot 10 + 5 \cdot 2$</td>
</tr>
<tr>
<td>$a + b = b + a$</td>
<td>$x \cdot y = y \cdot x$</td>
<td>$(3 + 4) + 6 = 3 + (4 + 6)$</td>
</tr>
<tr>
<td>$2 + 3 = 3 + 2$</td>
<td>$(j \cdot k) \cdot l = j \cdot (k \cdot l)$</td>
<td>$(l + m) + n = l + (m + n)$</td>
</tr>
</tbody>
</table>

**LESSON 4:** Are They Saying the Same Thing? • M3-63
Assignment

Write
Match each term to the best definition.
1. Commutative Property of Addition
   a. For any numbers \( a \) and \( b \), \( a + b = b + a \)
2. Commutative Property of Multiplication
   b. For any numbers \( a \), \( b \), and \( c \), \((ab)c = a(bc)\)
3. Associative Property of Addition
   c. Two or more terms that have the same variable raised to the same power.
4. Associative Property of Multiplication
   d. For any numbers \( a \) and \( b \), \( ab = ba \)
5. like terms
   e. For any numbers \( a \), \( b \), and \( c \), \((a + b) + c = a + (b + c)\)

6. Explain what it means for two expressions to be equivalent.

Remember
To determine whether two expressions are equivalent, you can create a table of values, graph the expressions, or rewrite the expressions using number properties.

Practice
Determine whether the two expressions are equivalent. Use properties, a table, and a graph in each problem to verify your answer.
1. \( 2(3x + 2) - 2x \) and \( 4x + 2 \)
2. \( 1 + 3(3 + x) \) and \( 4(3 + x) \)
3. \( 2x + 1 \) and \( 2(x + \frac{1}{2}) \)
4. \( \frac{(6x + 9)}{3} + 4 \) and \( 2(x + 3.5) \)

Stretch
Determine whether the two expressions are equivalent. Use properties, a table, and a graph in each problem to verify your answer.
1. \((x + 5)(2x + 1)\) and \(2x^2 + 5\)
2. \((x + 1)(x - 1)\) and \(x^2 - 1\)
Review

Use the Distributive Property and combine like terms to rewrite each expression.

1. $9(6m + 3) + 6(1 - 4m)$
2. $\frac{3(4x + 8y)}{6} + 2y - x$

Determine the better buy.

3. 6 car washes for $50 or 4 car washes for $36
4. 10 markers for $2.40 or 32 markers for $7.00

Determine the least common multiple (LCM) of each pair of numbers.

5. 6 and 10
6. 7 and 12
LESSON 5: DVDs and Songs

Using Algebraic Expressions to Analyze and Solve Problems

WARM UP
Blake is twice as old as Alec.
Celia is 3 years older than Blake.
1. If Alec is 9 years old, how old is Blake?
2. If Alec is 9 years old, how old is Celia?
3. If Celia is 13 years old, how old is Blake?
4. If Celia is 13 years old, how old is Alec?
5. If Blake is 30 years old, how old is Alec?
6. If Blake is 30 years old, how old is Celia?

LEARNING GOALS
• Represent real-world problems with algebraic expressions.
• Use variables and write algebraic expressions to solve real-world and mathematical problems.

You have written numeric and algebraic expressions. How can algebraic expressions help you solve real-world problems?
Number Magic

Complete the number riddle by following each step.

Step 1: Pick a number between 1 and 30.
Step 2: Add 9 to your number.
Step 3: Multiply the sum by 3.
Step 4: Subtract 6 from the product.
Step 5: Divide the difference by 3.
Step 6: Subtract your original number.

1. Record your answer.

2. Compare your original number and your result with a classmate’s number and result.

3. Use properties of numbers to demonstrate why the riddle works.

Haley says: “I have twice as many DVDs as Jaret.”

Dillan says: “I have four more DVDs than Haley.”

Kierstin says: “I have three times as many as Dillan.”

1. If Jaret has 10 DVDs, determine the number of DVDs for each friend. Explain your reasoning.

<table>
<thead>
<tr>
<th>Haley</th>
<th>Dillan</th>
<th>Kierstin</th>
<th>All four friends together</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. If Kierstin has 24 DVDs, determine the number of DVDs for each friend. Explain your reasoning.

<table>
<thead>
<tr>
<th>Haley</th>
<th>Dillan</th>
<th>Jaret</th>
<th>All four friends together</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. Let $j$ represent the number of DVDs that Jaret has.
   
a. Write an algebraic expression that represents the number of DVDs for each friend.
   
   Haley
   Dillan
   
   Kierstin
   
   All four friends together

b. Use your expression from Question 3, part (a), to determine the number of DVDs they have altogether if Jaret has:

   10 DVDs.
   2 DVDs.
   
   25 DVDs.
   101 DVDs.

   c. Write an algebraic expression to represent the number of DVDs for:
   
   Jaret and Dillan
   Haley and Kierstin

4. Let $k$ represent the number of DVDs Kierstin has.
   
a. Write an algebraic expression that represents the number of DVDs for each friend.
   
   Haley
   Dillan
   
   Jaret
   
   All four friends together
b. Use your expression from Question 6, part (a), to determine the number of DVDs they have altogether if Kierstin has:

- 72 DVDs.
- 24 DVDs.
- 36 DVDs.
- 660 DVDs.

c. Write an algebraic expression to represent the number of DVDs for:

- Jaret and Dillan
- Haley and Kierstin

5. Let $h$ represent the number of DVDs Haley has.

a. Write an algebraic expression that represents the number of DVDs for each friend.

- Jaret
- Dillan
- Kierstin
- All four friends together

b. Use your expression from Question 9, part (a), to determine the number of DVDs they have altogether if Haley has:

- 20 DVDs.
- 24 DVDs.
- 50 DVDs.
- 34 DVDs.
c. Write an algebraic expression to represent the number of DVDs for:

Jaret and Dillan  Haley and Kierstin

6. Let $d$ represent the number of DVDs Dillan has.

a. Write an algebraic expression that represents the number of DVDs for each friend.

Jaret  Haley

Kierstin  All four friends together

b. Use your expression from Question 11, part (a), to determine the number of DVDs they have altogether if Dillan has:

24 DVDs.  8 DVDs.

20 DVDs.  60 DVDs.

c. Write an algebraic expression to represent the number of DVDs for:

Jaret and Dillan  Haley and Kierstin
Five friends have their own MP3 players.

Jake has 5 more songs on his MP3 than Rick has on his.

Marilyn has half as many songs on her MP3 as Jake has on his.

Lori has 3 more than twice the number of songs on her MP3 as Rick has on his.

Cody has 3 times as many songs on his MP3 as Marilyn has on hers.

1. Let $r$ represent the number of songs on Rick’s MP3 player.
   Write an algebraic expression that represents the number of songs on each friend’s MP3 player.

   Jake
   Marilyn

   Lori
   Cody
   All five friends together

2. Use your expression from Question 1 to calculate the number of songs they have altogether if Rick has:
   a. 15 songs.
   b. 47 songs.

3. Write an algebraic expression to represent the number of songs for:
   a. Jake, Cody, and Rick
   b. Marilyn and Lori
TALK the TALK

Be a Magician!

You started this lesson by looking at a number riddle. Now that you have explored algebraic expressions, you can think about how they work.

1. Write the corresponding algebraic expressions for each step to show why this number trick works.

   - Choose a number.
   - Add 5.
   - Double the result.
   - Subtract 4.
   - Divide the result by 2.
   - Subtract the number you started with.
   - The result is 3.

2. Create your own number trick. Then write the corresponding algebraic expressions to show why it works.
Practice
At the end of each school year, Evan cleans out all of the school supplies that have collected in his desk. He can’t believe how much stuff is in there this year! He has 4 times as many markers as he has pencils. He has 3 more highlighters than he has markers. He has twice as many pens as he has highlighters.

1. Suppose Evan found 5 pencils in his desk.
   a. Determine the number of markers that are in his desk. Explain your reasoning.
   b. Determine the number of highlighters that are in his desk. Explain your reasoning.
   c. Determine the number of pens that are in his desk. Explain your reasoning.
   d. Determine the total number of writing utensils that are in his desk. Explain your reasoning.

2. Suppose Evan found 78 pens in his desk.
   a. Determine the number of highlighters that are in his desk. Explain your reasoning.
   b. Determine the number of markers that are in his desk. Explain your reasoning.
   c. Determine the number of pencils that are in his desk. Explain your reasoning.
   d. Determine the total number of writing utensils that are in his desk. Explain your reasoning.

3. Let \( p \) represent the number of pencils that Evan has in his desk.
   a. Write an algebraic expression that represents the number of markers in Evan’s desk.
   b. Write an algebraic expression that represents the number of highlighters in Evan’s desk.
   c. Write an algebraic expression that represents the number of pens in Evan’s desk.
   d. Write an algebraic expression that represents the total number of writing utensils in Evan’s desk.
   e. Use your expression from part (d) to determine the total number of writing utensils in Evan’s desk if there are 8 pencils.
   f. Use your expression to determine the total number of writing utensils in Evan’s desk if there are 12 pencils.

Write
How can algebraic expressions help you to solve real-world problems?

Remember
An algebraic expression is a mathematical phrase involving at least one variable and sometimes numbers and operation symbols.
Write an algebraic expression to represent each verbal expression.
1. One-third the sum of a number and two and one hundredths.
2. Sixteen and two-tenths subtracted from two times a number.

Calculate each percent.
3. In Ms. Romano’s math class of 25 students, 8 of the students play a musical instrument. What percent of the class plays a musical instrument?
4. In Ms. Sobato’s science class of 20 students, 3 of the students are in the school play. What percent of the class is in the school play?

Determine each whole for the percent and part given.
5. 68 is 32% of what number?
6. 16 is 80% of what number?
Expressions Summary

KEY TERMS
- power
- base
- exponent
- perfect square
- perfect cube
- evaluate a numeric expression
- Order of Operations
- variable
- algebraic expression
- coefficient
- term
- evaluate an algebraic expression
- like terms
- Distributive Property
- equivalent expressions

Repeated multiplication can be represented as a power. A power has two elements: the base and the exponent. The base of a power is the factor that is multiplied repeatedly in the power, and the exponent of the power is the number of times the base is used as a factor.

You can read this power in different ways: “2 to the fourth power,” “2 raised to the fourth power,” or “2 to the fourth.”

A number multiplied by itself is a square. The squares of integers are called perfect squares. For example, 9 is a perfect square because \(3 \times 3 = 9\). Another way to write this equation is \(3^2 = 9\). You can read \(3^2\) as “3 squared.”

A number used as a factor three times is a cube. A perfect cube is the cube of an integer. For example, 216 is a perfect cube because \(6 \times 6 \times 6\), or \(6^3\), is equal to 216. You can read \(6^3\) as “6 cubed.”
To **evaluate a numeric expression** means to simplify the expression to a single numeric value.

For example, consider the numeric expression $2 \cdot 5^2$ represented by the model shown.

$$5^2 = 25,$$
$$2 \cdot 5^2 = 2 \cdot 25,$$
$$50.$$

Therefore, $2 \cdot 5^2$ has a value of 50.

The **Order of Operations** is a set of rules that ensures the same result every time an expression is evaluated.

1. Evaluate expressions inside parentheses or grouping symbols.
2. Evaluate exponents.
3. Multiply and divide from left to right.
4. Add and subtract from left to right.

For example, evaluate the expression $12 \div (4 + 2) + 4^2$ using the Order of Operations.

$$12 \div (6) + 4^2$$
**Evaluate the expression in parentheses.**

$$12 \div 6 + 16$$
**Evaluate the exponent.**

$$2 + 16$$
**Divide from left to right.**

$$18$$
**Add from left to right.**

In mathematics, letters are often used to represent quantities that vary. These letters are called **variables**, and they help you write algebraic expressions to represent situations. An **algebraic expression** is an expression that has at least one variable.

For example, if a school lunch costs $2.25 for each student, you can write an algebraic expression to represent the total amount of money collected for any number of students buying school lunches.

The variable $s$ can represent the unknown number of students buying school lunches. The algebraic expression is $2.25s$.

A number that is multiplied by a variable in an algebraic expression is called the numerical **coefficient**. The coefficient in the expression written above is 2.25.
A term of an algebraic expression is a number, variable, or product of numbers and variables.

For example, consider the expression $3x + 4y - 7$. The expression has three terms: $3x$, $4y$, and $7$. The operation between the first two terms is addition, and the operation between the second and third term is subtraction. There are two terms with variables and the third term is a constant term of $7$.

To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable. When you evaluate an algebraic expression, you substitute the given values for the variables, and then determine the value of the expression.

For example, evaluate $10 - \frac{x}{3}$, for $x = 9$.

$$10 - \frac{9}{3} \quad \text{Substitute the given value for } x.$$ 
$$10 - 3 = 7 \quad \text{Use the Order of Operations to evaluate the expression.}$$

In an algebraic expression, like terms are two or more terms that have the same variable raised to the same power. The numerical coefficients of like terms may be different. You can combine like terms in algebraic expressions by adding or subtracting terms with the same variables. For example, $3x + 5x$ combines to make $8x$.

Algebraic expressions can be rewritten using the Distributive Property.

For example, consider the expression $5(x + 1)$, which has two factors: $5$ and the quantity $(x + 1)$. In this case, multiply the $5$ by each term of the quantity $(x + 1)$. The model using algebra tiles is shown.

$$5(x + 1) = 5x + 5$$
You can also rewrite an expression of the form \( \frac{4x + 8}{4} \) using the Distributive Property.

The Distributive Property can also be used to rewrite an algebraic expression as a product of two factors: a constant and a sum of terms. This is also referred to as factoring expressions.

To rewrite the expression \( 4x - 7 \) so the coefficient of the variable is 1, factor out the coefficient 4 from each term. The equivalent expression is the product of the coefficient and the sum of the remaining factors.

\[
4x - 7 = 4 \left( \frac{4x}{4} - \frac{7}{4} \right) = 4 \left( x - \frac{7}{4} \right)
\]

**LESSON 4**

**Are They Saying the Same Thing?**

Two algebraic expressions are **equivalent expressions** if, when any values are substituted for the variables, the results are equal.

Creating a table of values, graphing the expressions, or rewriting the expressions using number properties can help you to determine if two expressions are equivalent.

For example, consider the two expressions \( 3(x + 1) - 2x \) and \( x + 3 \).

The table of values for each value of \( x \) shows that the two expressions are equivalent.

Using the table of values to graph the two expressions results in two lines that lie on top of one another, showing that the expressions are equivalent.
You can also rewrite the given expression to show that the two expressions are equivalent.

\[ 3(x + 1) - 2x \quad \text{Given} \]
\[ 3x + 3 - 2x \quad \text{Distributive Property} \]
\[ x + 3 \quad \text{Combine Like Terms} \]

You can use algebraic expressions to represent, analyze, and solve real-world problems. For example, let \( k \) represent the number of books that Karen has.

Jack has three times as many books as Karen: \( 3k \).

Daniel has 6 more books than Jack: \( 3k + 6 \).

Hannah has twice as many books as Daniel: \( 2(3k + 6) \).

If Karen has 10 books, determine the total number of books the four friends have together.

Substitute 10 for \( k \) in each expression to determine the number of books each friend has.

- Jack has \( 3 \cdot 10 \), or 30 books.
- Daniel has \( 3 \cdot 10 + 6 \), or 36 books.
- Hannah has \( 2(3 \cdot 10 + 6) \), or 72 books.

The total number of books the four friends have together is \( 10 + 30 + 36 + 72 = 148 \) books.
In a tug-of-war contest, one side may be stronger, so the two sides would be unequal. Or both sides could be equally strong. No matter what, these two puppies = cute.

Lesson 1
First Among Equals
Reasoning with Equal Expressions ................................................. M3-87

Lesson 2
Bar None
Solving One-Step Addition Equations ............................................ M3-107

Lesson 3
Play It In Reverse
Solving One-Step Multiplication Equations ..................................... M3-119

Lesson 4
Getting Real
Solving Equations to Solve Problems ............................................. M3-135
TOPIC 2: EQUATIONS

In this topic, students use their understanding of expressions to create equations and determine if given values are solutions to equations or simple inequalities. Students develop an understanding of the equals sign as indicating a relationship, not as an operator. They learn that solving an equation means maintaining equality of the expressions on either side of the equals sign. Students analyze equations and generalize that equations can have one solution, no solution, or infinitely many solutions. Students use bar models to reason about solving one-step addition and multiplication equations. Bar models are used to emphasize the importance of maintaining equality. Students solve a variety of real-world problems by writing and solving one-step addition and multiplication equations.

Where have we been?

In the previous topic, students learned about expressions. Instruction in this topic asks students to set expressions equal to each other and to determine if they are equal. Students have been solving for missing values in addition and subtraction equations since grade 1 and in multiplication and division equations since grade 3. However, this topic is likely the first time they have used variables to represent and solve for an unknown value in an equation.

Where are we going?

In grade 7, students will expand their ability to solve equations to two-step linear equations. By grade 8, students will be able to solve a wide variety of linear equations. In high school, students will expand their skills to include solving exponential, quadratic, polynomial, and trigonometric equations. All of this work is built upon the foundation of equivalent expressions that students begin to build in this topic.

Using Bar Models to Solve Addition Equations

Bar models are visual tools that can be used to solve equations. For example, this bar model shows that the expressions $x + 10$ and 15 are equal, so $x + 10 = 15$. The top bar can be split into two bars, $x$ and 10. When this split happens in the bottom bar, with one bar containing 10, it shows that $x$ is the same as 5, so $x = 5$. 

<table>
<thead>
<tr>
<th>x</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + 10$</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>
Myth: “Just give me the rule. If I know the rule, then I understand the math.”

Memorize the following rule: All quars are elos. Will you remember that rule tomorrow? Nope. Why not? Because it has no meaning. It isn’t connected to anything you know. What if we change the rule to: All squares are parallelograms. How about now? Can you remember that? Of course you can because now it makes sense.

Learning does not take place in a vacuum. It must be connected to what you already know. Otherwise, arbitrary rules will be forgotten.

#mathmythbusted

Talking Points
You can further support your student’s learning by making sure they eat right and get enough sleep. Healthy bodies make for healthy minds, and both diet and sleep have significant effects on learning.

Key Terms

**equation**
An equation is a mathematical sentence that contains an equals sign. An equation can contain numbers, variables, or both in the same mathematical sentence.

**inverse operations**
Inverse operations are pairs of operations that undo the effects of each other.
WARM UP

Rewrite each number as an addition, subtraction, multiplication, or division expression. Use each operation once.

1. 24
2. $\frac{1}{2}$
3. 0
4. 100

LEARNING GOALS

- Compose and decompose numeric and algebraic equations.
- Substitute values into equations to determine whether they make the equation true.
- Construct and analyze equations using Properties of Equality.
- Analyze, write, and graph inequalities.
- Determine the number of solutions of an equation or inequality.

KEY TERMS

- equation
- Reflexive Property of Equality
- solution
- Addition Property of Equality
- Subtraction Property of Equality
- Multiplication Property of Equality
- Division Property of Equality
- Symmetric Property of Equality
- Zero Property of Multiplication
- Identity Property of Multiplication
- Identity Property of Addition
- graph of an inequality
- solution set of an inequality

You have learned about both numeric and algebraic expressions and how they describe situations and relationships among quantities. What properties do equal expressions have and how can you use these properties to reason about solutions?

LESSON 1: First Among Equals • M3-87
The Same But Different

1. Write different expressions equal to 4.

\[ \_\_\_\_\_\_ = 4 \]

4 = \_\_\_\_\_

4 = \_\_\_\_\_

2. Now write different expressions equal to 4 + 5.

\[ 4 + 5 = \_\_\_\_\_\_\_\_\_\_ \]

\[ \_\_\_\_\_\_\_\_\_\_ = 4 + 5 \]

4 + 5 = \_\_\_\_\_\_\_\_\_\_

3. What can you do to one of the expressions you wrote in Question 1 to make it equal to one of the expressions you wrote in Question 2?

Be creative! Include different operations in your expressions.
An equation is a statement of equality between two expressions. An equation can contain numbers, variables, or both in the same mathematical sentence.

Consider the equation $8 + 4 = \_ + 5$. It has an unknown number.

One way to determine the unknown number is to rewrite the expressions on both sides of the equals sign until they match.

Consider each reasoning strategy that is used to determine the unknown number in $8 + 4 = \_ + 5$.

**Rylee**
The equal sign tells me to perform the operation on the left in the equation $8 + 4 = \_ + 5$.

\[
8 + 4 = 12 + 5 \\
12 + 5 = 17 
\]

Therefore, the unknown number is 17.

**Clover**
I can determine the unknown number in $8 + 4 = \_ + 5$ by rewriting the expression on the left. I can take 1 from 8 and give it to the 4 and keep the value of the expression the same.

\[
(8 - 1) + (4 + 1) = \_ + 5 \\
7 + 5 = \_ + 5 
\]

Therefore, the unknown number is 7.
1. What is the unknown number in the equation \(8 + 4 = \_ + 5\)? Explain how this makes sense.

2. Explain the error in Rylee's reasoning.

3. How are Clover's reasoning and Fiona's reasoning similar? How are they different?
4. Consider the equation $31 + 67 = \_ + 12$.

   a. Determine the unknown number by rewriting the expressions on either side of the equals sign until they match.

   b. How can you check your answer to make sure it is correct?

   c. What number property or properties did you use when determining the unknown number?

5. Use your number sense reasoning to determine each unknown number. Show your work.

   a. $85 + 45 = \_ + 60$  
   b. $9 + 23 = \_ + 14$
Equations come in many forms. Because expressions are either numeric or algebraic, equations can be made of just numbers or both numbers and variables.

Equations are statements—they may be always true, never true, or true only for one or more values of the variable.

<table>
<thead>
<tr>
<th>Always True</th>
<th>Never True</th>
<th>True for certain values of the variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 = 10 - 4$</td>
<td>$10 = 20$</td>
<td>$x = 5$</td>
</tr>
<tr>
<td>$x = x$</td>
<td>$x = x + 2$</td>
<td>$x + 2 = 12$</td>
</tr>
</tbody>
</table>

When you determine that an equation is never true, you can make it a true statement by using the symbol $\neq$. For example, $10 = 20$ should be written as $10 \neq 20$.

1. Create at least five different kinds of equations using the list of expressions given.

2. Identify your equations that are always true, never true, and those equations where you don’t yet know whether they are true or false. Explain your reasoning.
A solution to an equation is any value for a variable that makes the equation true.

3. Sets of values are given. For each set, decide which value(s), if any, makes each of your equations from Question 1 true. Show your work.
   a. \{1, 2, 3, 4\}
   b. \{1, 3, 5, 7, 9\}
   c. \{0\}

4. Use the list of given expressions to write the type of equation described.
   a. Write an equation with variables that has no possible solution. Explain why the equation has no solution.
   b. Write an equation with variables that is true no matter what number is substituted for the variable. Explain why there are an infinite number of solutions.
The **Addition Property of Equality** states that if two values \( a \) and \( b \) are equal, when you add the same value \( c \) to each, the sums are equal.

The **Subtraction Property of Equality** states that when you subtract the same value \( c \) from equal values \( a \) and \( b \), the differences are equal.

<table>
<thead>
<tr>
<th>Properties of Equality</th>
<th>For all numbers ( a, b, ) and ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition Property of Equality</td>
<td>If ( a = b ), then ( a + c = b + c ).</td>
</tr>
<tr>
<td>Subtraction Property of Equality</td>
<td>If ( a = b ), then ( a - c = b - c ).</td>
</tr>
</tbody>
</table>

1. **Suppose you have the equation** \( x = 15 \).

   a. Use the **Addition Property of Equality** to write at least 3 equations that have the same solution.

   b. Use the **Subtraction Property of Equality** to write at least 3 equations that have the same solution.

2. **Suppose you have the equation** \( x + 5 = 1 + 9 \).

   a. Use the **Addition Property of Equality** to write at least 3 equations that have the same solution.

   b. Use the **Subtraction Property of Equality** to write at least 3 equations that have the same solution.
The **Multiplication Property of Equality** states that if two values \( a \) and \( b \) are equal, when you multiply each by the same value \( c \), the products are equal. The **Division Property of Equality** states that when you divide equal values \( a \) and \( b \) by the same value \( c \), the quotients are equal. The Division Property of Equality is true only if \( c \) is not equal to 0.

<table>
<thead>
<tr>
<th>Properties of Equality</th>
<th>For all numbers ( a ), ( b ), and ( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplication Property of Equality</td>
<td>If ( a = b ), then ( a \cdot c = b \cdot c ).</td>
</tr>
<tr>
<td>Division Property of Equality</td>
<td>If ( a = b ) and ( c \neq 0 ), then ( a \div c = b \div c ).</td>
</tr>
</tbody>
</table>

3. Suppose you have the equation \( x = 5 \).

   a. Use the Multiplication Property of Equality to write at least 3 equations that have the same solution.

   b. Use the Division Property of Equality to write at least 3 equations that have the same solution.

4. Suppose you have the equation \( \frac{1}{2}x = 10 \).

   a. Use the Multiplication Property of Equality to write at least 3 equations that have the same solution.

   b. Use the Division Property of Equality to write at least 3 equations that have the same solution.

5. Describe how you can check the solutions of the equations you wrote in Questions 1 and 3.
Cut out the cards at the end of the lesson. There are Equation Cards and Solution Cards. The Solution Cards are shaded blue.

1. Match the Equation Cards with the Solution Cards. Explain how you identified each solution.

2. Which equation(s) have no solutions? Explain how you know.

3. Which equation(s) have an infinite number of solutions? Explain how you know.

The Symmetric Property of Equality states that if \( a = b \), then \( b = a \). So, \( x = 3 \) is the same as \( 3 = x \).
Equations that have an infinite number of solutions are equations that are true no matter what value you assign to the variable. These kinds of equations often describe important properties of numbers.

Consider each property.

- The **Zero Property of Multiplication** states that the product of any number and 0 is 0.
- The **Identity Property of Multiplication** states that the product of any number and 1 is the number.
- The **Identity Property of Addition** states that the sum of any number and 0 is the number.

4. **Study the Equation Cards.**

   a. Which equation(s) states the Zero Property of Multiplication?

   b. Which equation(s) states the Identity Property of Multiplication?

   c. Which equation(s) states the Identity Property of Addition?

5. Three of the Solution Cards did not match any of the Equation Cards. Write equations that have those values as solutions.
You can use a number line to represent inequalities. The **graph of an inequality** in one variable is the set of all points on a number line that make the inequality true. The set of all points that make an inequality true is the **solution set of the inequality**.

1. Consider the graphs of the inequalities $x > 3$ and $x \geq 3$.

   ![Graphs of inequalities](image)

   **a.** Describe each number line representation.

   **b.** Describe the solution set for each inequality.

   **c.** How does the solution set of the inequality $x \geq 3$ differ from the solution set of $x > 3$?
2. Consider the graphs of the inequalities \( x < 3 \) and \( x \leq 3 \).

\[
\begin{align*}
x < 3 & \quad \text{a. Describe each number line representation.} \\
x \leq 3 & \\
\end{align*}
\]

b. Describe the solution set for each.

c. How does the solution set of the inequality \( x \leq 3 \) differ from the solution set of \( x < 3 \)?

The solution to any inequality can be represented on a number line by a ray. A ray begins at a starting point and goes on forever in one direction.

A closed circle means that the starting point is part of the solution set of the inequality. An open circle means that the starting point is not part of the solution set of the inequality.
3. Write the inequality represented by each graph.

a. $x \leq 14$

b. $x < 55$

c. $2\frac{1}{2} \leq x$

d. $x > 3.3$

e. $x \neq 4.2$
5. Consider the inequalities in Questions 1 through 4.

a. How many solutions does each inequality have?

b. Can you write an inequality that has no solutions? Explain.

c. Can you write an inequality that has just one solution? Explain your reasoning.

6. Explain the meaning of each sentence in words. Then, define a variable and write a mathematical statement to represent each statement. Finally, sketch a graph of each inequality.

a. The maximum load for an elevator is 2900 lbs.

b. A car can seat up to 8 passengers.

c. No persons under the age of 18 are permitted.

d. You must be at least 13 years old to join.
TALK the TALK

Not All Variables Are Created Equal

A point at $a$ is plotted on the number line shown.

1. Plot a point to the right of this point and label it $b$.
   a. Write three different inequalities that are true about $a$ and $b$.
   b. What can you say about all points to the right of point $a$ on the number line?

2. Plot a point to the left of $a$ and label it $c$.
   a. Write three different inequalities that are true about $a$ and $c$.
   b. What can you say about all the points to the left of point $a$ on the number line?

3. Describe the position of all the points on the number line that are:
   a. greater than $a$.  
   b. less than $a$.  

**Equation Cards**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
<th>Equation</th>
<th>Solution</th>
<th>Equation</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$11 + x = 11$</td>
<td>$x = 0$</td>
<td>$10x = 30$</td>
<td>$x = 3$</td>
<td>$x + 3.5 = 14.25$</td>
<td>$x = 4.25$</td>
</tr>
<tr>
<td>$1x + 6 = 9$</td>
<td>$x = 0$</td>
<td>$5/8 = x + 1/2$</td>
<td>$x = 4$</td>
<td>$1/8x = 8$</td>
<td>$x = 32$</td>
</tr>
<tr>
<td>$5 = x + 5$</td>
<td>$x = 0$</td>
<td>$29 = x \cdot 29$</td>
<td>$x = 100$</td>
<td>$x = 5$</td>
<td>$x = 1$</td>
</tr>
<tr>
<td>$x \cdot 1 = x$</td>
<td>$x = 0$</td>
<td>$x + 0 = x$</td>
<td>$x = 0$</td>
<td>$x = 1/4x$</td>
<td>$x = 0$</td>
</tr>
<tr>
<td>$x + 4 = 4 + x$</td>
<td>$x = 0$</td>
<td>$x/4 = 1/4x$</td>
<td>$x = 0$</td>
<td>$x = 0.1$</td>
<td>$x = 0.1$</td>
</tr>
</tbody>
</table>

**Solution Cards**

| $x = 10.75$    | $x = 0$      | $3 = x$        | no solutions | $x = 1/8$      |
| $x = 18$       | $x = 0$      | $x = 100$      | $x = 1$      | $6 = x$        |
| $9 = x$ (nine) | $x = 1/4$    | $64 = x$       | infinite solutions | $x = 0.1$      |
Assignment

Write
Complete each statement with the correct term.

1. The ______________ states that if two values a and b are equal, when you multiply each by the same value c, the products are equal.
2. A ______________ to an equation is any value for a variable that makes the equation true.
3. The ______________ says that when both sides of an equation look exactly the same, their values must be equal.
4. An ______________ is a mathematical sentence created by writing two expressions with an equals sign between them.
5. The ______________ is the set of all points on a number line that make the inequality true.

Remember
Properties of Equality are logical rules that allow you to maintain balance and rewrite equations.

<table>
<thead>
<tr>
<th>Properties of Equality</th>
<th>For all numbers a, b, and c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition Property of Equality</td>
<td>If a = b, then a + c = b + c.</td>
</tr>
<tr>
<td>Subtraction Property of Equality</td>
<td>If a = b, then a − c = b − c.</td>
</tr>
<tr>
<td>Multiplication Property of Equality</td>
<td>If a = b, then ac = bc.</td>
</tr>
<tr>
<td>Division Property of Equality</td>
<td>If a = b, and c ≠ 0, then ( \frac{a}{c} = \frac{b}{c} ).</td>
</tr>
</tbody>
</table>

Practice
Indicate whether each equation has one solution, no solutions, or an infinite number of solutions and explain your reasoning. If the equation has one solution, determine the solution from the set of values given.

Set of values: \{0, 1, 2, 3, 4, 5, 9, 10, 35, 36, 37, 38, 39, 40, 50, 60, 61, 62, 63, 64, 65, 99, 100\}

1. \( x - 3 = x + 3 \)  
2. \( 4 \cdot x = 20 \)  
3. \( 81 = 9x \)  
4. \( x + 17 = 55 \)  
5. \( \frac{x}{3} = 21 \)  
6. \( 1x = x \)  
7. \( 8 + x = x + 8 \)  
8. \( 99 = x - 1 \)
Stretch
Model each equality or inequality situation. Then determine each solution.

1. Najid is taller than Emily and shorter than Daniel. Who is the tallest?
2. Sophie is now as old as Jasmine was 6 years ago. Who is older?

Review
1. Define variables and write an algebraic expression to represent each situation.
   a. Miguel has three times as many books as Jose.
   b. Rosa has 5 fewer bracelets than Maria.

For each situation, examine the given expressions and/or solution strategies.

2. Darian’s band made $500 on one night. They had to subtract costs of $80 and then divide the remaining money among the band members. If there are 4 members in the band, which numeric expression correctly shows the amount that each member will make? Explain your answer using the rules for order of operations.

Expression A
500 – 80 ÷ 4

Expression B
(500 – 80) ÷ 4

3. Darian’s band hires a manager. The manager asks a local park if they can hold a concert on one of the lawn areas. The lawn can have 20 rows of seats with 20 chairs in each row. The band charges $25 for each seat. The cost for advertising, the rental of the chairs, and the management fees totals $4000. If the band is able to fill all of the seats, which solution shows the amount the band will make? Determine the error that was made in the incorrect solution.

Solution A
25 × 20^2 – 4000
25 × 400 – 4000
10,000 – 4000
6000

Solution B
25 × 20^2 – 4000
500^2 – 4000
250,000 – 4000
246,000

4. Determine each unknown.
   a. \[ \frac{5}{6} = \frac{x}{30} \]
   b. \[ \frac{90}{x} = \frac{9}{2} \]
LEARNING GOALS
- Reason about addition equations.
- Use bar models to represent one-step addition equations.
- Use inverse operations to solve one-step addition equations.
- Solve one-step addition equations.

KEY TERMS
- bar model
- one-step equation
- inverse operations

Throughout this course, you have used a variety of tools to solve mathematical problems, including area models, pictures, tables, tape diagrams, double number lines, graphs, and expressions. What tools might help you in solving equations?

WARM UP
Determine each sum or difference.

1. $5.67 + 8.73$
2. $8.73 - 5.67$
3. $\frac{3}{7} + \frac{4}{5}$
4. $\frac{20}{3} - \frac{15}{4}$
Form of 0

Consider the number 0. What comes to mind?

1. Write five different numeric expressions for the number 0.

Be creative! Use different types of numbers and operations in your expressions.

Share your numeric expressions with your classmates.

2. Did you and your classmates use common strategies to write your expressions? How many possible numeric expressions could you write for this number?
Reasoning about equations and determining solutions with bar models provides a visual representation of the structure of the equations. A bar model uses rectangular bars to represent known and unknown quantities.

**WORKED EXAMPLE**

Consider the addition equation \( x + 10 = 15 \).

This equation states that for some value of \( x \), the expression \( x + 10 \) is equal to 15. This can be represented using a bar model.

Just like with area models, bar models can be decomposed. The expression \( x + 10 \) can be decomposed into a part representing \( x \) and a part representing 10. The number 15 can be decomposed in a similar way: \( 15 = 5 + 10 \).

The bar model demonstrates that these two equations are equivalent.

\[
\begin{align*}
x + 10 &= 15 \\
x + 10 &= 5 + 10
\end{align*}
\]

By examining the structure of the second equation, you can see that 5 is the value for \( x \) that makes this equation true.
1. Why is the number 15 decomposed into the numeric expression $5 + 10$?

2. Describe how the model in the worked example would be different for each equation. Complete the bar model for each.
   a. $x + 10 = 17$
   b. $x + 6 = 15$

3. Consider the equation $14 + x = 32$.
   a. Complete the bar model.
   b. Write the equation represented by the decomposed expressions in the bar model.
   c. Which value for $x$ makes the equation a true statement?
4. Consider the equation \( 90 = x + 64 \).
   a. Complete the bar model.

   b. Write the expression represented by the decomposed expressions in the bar model.

   c. Which value for \( x \) makes the equation a true statement?

5. In each bar model, how did you determine how to decompose the given expressions?
In Activity 2.1, *Reasoning About Equations*, you used bar models to solve one-step equations. A one-step equation is an equation that can be solved using only one operation. How can you use what you learned from creating bar models to solve any equation?

Now that you understand the bar model, you can write equivalent equations with the same structure. While you can use reasoning to determine the value for the variable that makes an equation true, you can also use the properties and inverse operations to isolate the variable. Inverse operations are pairs of operations that reverse the effects of each other. For example, subtraction and addition are inverse operations.

**WORKED EXAMPLE**

Solve the equation $h + 6 = 19$.

\[
\begin{align*}
    h + 6 & = 13 + 6 & \text{Write equivalent expressions that mirror structure.} \\
    h + 6 - 6 & = 13 + 6 - 6 & \text{Use inverse operations to reverse the addition of 6 to } h. \\
    h + 0 & = 13 + 0 & \text{Combine like terms and apply the Additive Identity Property.} \\
    h & = 13
\end{align*}
\]

1. Examine the worked example.
   a. What is the solution to $h + 6 = 19$?

   b. Are there other solutions to the equation? How do you know?
2. Use the same strategy to solve each equation.
   a. $35 = 12 + m$
   b. $t + 24 = 85$

3. Analyze Kaniah's strategy to solve the equation $11 = m + 7$.

   **Kaniah**
   When solving the addition equation $11 = m + 7$, I can simply subtract 7 from both sides without first writing an equivalent equation.

   
   \[
   \begin{align*}
   11 &= m + 7 \\
   11 - 7 &= m + 7 - 7 \\
   4 &= m
   \end{align*}
   \]

   The value for $m$ that makes this equation true is 4.

   a. What Property of Equality is Kaniah using in her strategy?

   b. How could Kaniah check that her solution is correct?
4. Use Kaniah’s strategy to solve each equation. Check to see that your solution makes the original equation a true statement.

a. $120 + y = 315$   
b. $\frac{5}{4} = x + 4\frac{1}{2}$

c. $b + 5.67 = 12.89$   
d. $2356 = a + 1699$

e. $\frac{7}{12} = g + \frac{1}{4}$   
f. $w + 3.14 = 27$

g. $13\frac{7}{8} = c + 9\frac{3}{4}$   
h. $19 + p = 105$
1. Braeden thinks that he can use decomposition to reason about more complicated equations, such as $4x = 20 + 3x$.

   Is Braeden correct? Show your work.

2. Think about each algebraic equation. Use reasoning to describe a relationship between $c$ and $d$ that makes the mathematical sentence true.

   a. $c + 23 = d + 14$
   b. $45 + c = 66 + d$

   c. $c + 3d = 2c$
   d. $4c + d + 10 = 8c + 2d$
TALK the TALK

It All Adds Up

1. What does it mean to solve an equation?

2. Describe how to solve any one-step addition equation. How do you check to see if a value is the solution to an equation?

3. Write two different one-step equations for each solution provided.
   a. \( m = 12 \)  
   b. \( 5 = x \)
   c. \( 5.6 = h \)  
   d. \( j = 6\frac{4}{7} \)
Assignment

Write
Write a definition for each term in your own words.
• one-step equation
• solution
• inverse operations

Remember
A solution to an equation is the value or values for the variable that makes the equation true. To solve a one-step addition equation, isolate the variable using number sense or the Subtraction Property of Equality.

Practice
Use a bar model to solve each equation.
1. \( x + 7 = 15 \)
2. \( 19 = x + 13 \)
3. \( 14.5 = 6 + y \)
4. \( a + \frac{1}{2} = 4\frac{3}{4} \)

Solve each equation. Check each solution.
5. \( 34 = x + 17 \)
6. \( a + 25 = 92 \)
7. \( 7\frac{3}{5} + b = 10\frac{3}{4} \)
8. \( 24\frac{1}{2} = t + 5\frac{1}{4} \)
9. \( r + 3.4 = 13.1 \)
10. \( 4.21 = 2.98 + s \)

Stretch
Solve each equation. Check each solution.
1. \( 34 = x - 17 \)
2. \( a - 25 = 92 \)
3. \( r - 3.4 = 13.1 \)
4. \( 24\frac{1}{2} = t - 5\frac{1}{4} \)

Review
Use the Properties of Equality to write 2 equations that have the given solution. Identify which property of equality was used.
1. \( j = 3 \)
2. \( 8 = m \)

Define variables and write an algebraic expression to represent each situation.
3. Terrance has one fewer sibling than Casey. Kolbie has three more siblings than Terrance.
4. Connor has half as many comic books as Devyn. Isaac has 4 more comic books than Connor.

Rewrite each expression.
5. \( \frac{2}{3}x + \frac{4}{5}x \)
6. \( \frac{1}{3}(\frac{2}{5}x) \)
LEARNING GOALS

- Use bar models to represent one-step multiplication equations.
- Use inverse operations to solve one-step multiplication equations.
- Reason about multiplication equations.
- Connect bar models to the algorithm for solving multiplication equations.
- Solve one-step multiplication equations.

You have solved one-step addition equations using bar models and inverse operations. How can you use similar strategies to solve one-step multiplication problems?
Form of 1

Consider the number 1. What comes to mind?

1. Write five different numeric expressions for the number 1.

Share your numeric expressions with your classmates.

2. Did you and your classmates use common strategies to write your expressions? How many possible numeric expressions could you write for this number?

Be sure to write expressions for 1 that include multiplication and division.
1. Why is the number 6 decomposed into the numeric expression 3 + 3?

Solve each equation using a bar model.

2. $3x = 12$
3. $7x = 63$
4. $4x = 6$

**WORKED EXAMPLE**

Consider the multiplication equation $2x = 6$.

This equation states that for some value of $x$, the expression $2x$ is equal to 6.

You can decompose $2x$ by rewriting it as the equivalent expression $1x + 1x$, or $x + x$.

To maintain equivalence, decompose 6 in a similar way.

The bar model demonstrates that these two equations are equivalent.

$$2x = 6$$

$$x + x = 3 + 3$$

By examining the structure of the second equation, you can see that $x = 3$. How do these bar models relate to the bar models you used to solve addition equations?
Multiplication equations often include numbers other than whole numbers.

Consider the equation \( \frac{1}{3}x = 2 \).

1. Explain how this equation compares to the equations in the previous activity.

WORKED EXAMPLE

Represent \( \frac{1}{3}x = 2 \) as a bar model.

To solve this equation for \( x \), compose 3 equally-sized parts to create the whole, \( x \).

To maintain equivalence, compose 3 equally-sized parts for the other expression, too.

This structure allows you to see the value of \( x \).

2. Complete the worked example by filling in the missing values. Then write the solution to the equation \( \frac{1}{3}x = 2 \).
Solve each equation using a bar model.

3. \( \frac{1}{4}x = 7 \)

4. \( \frac{x}{2} = 5 \)

5. Consider how to use bar models to solve \( \frac{2}{3}x = 8 \).
   Analyze each strategy.

Vanessa's Solution

```
\begin{array}{c|c|c|c}
\times & \frac{1}{3}x & \frac{1}{3}x & \frac{1}{3}x \\
\hline
\frac{2}{3}x & \frac{1}{3}x & & \\
\hline
8 & 4 & & \\
\hline
4 & 4 & 4 & 12
\end{array}
```

Carson's Solution

```
\begin{array}{c|c|c|c}
\times & \frac{2}{3}x & \frac{1}{3}x & \\
\hline
8 & 4 & & \\
\hline
12 & & & \\
\end{array}
```

This reminds me of tape diagrams.

a. How is Carson's solution strategy different from Vanessa's solution strategy?

b. What reasoning might Vanessa have used in her solution strategy?
Solve each equation using a bar model.

6. \( \frac{4}{5}x = 12 \)

7. \( \frac{3}{4}x = 8 \)

8. Consider the equation \( \frac{2}{3}x = 64 \).
   
   a. How does this fractional coefficient compare to the fractional coefficients that you have seen in this lesson so far?

   b. Create a bar model and solve for \( x \).

Reflect on the equations you solved in this activity.

9. How were they similar? What was common in how you used the bar models?
Consider the equation \( \frac{4}{5}x = \frac{1}{10} \).

1. How is this equation different from the equation you solved in the previous activity?

Compare the two solution strategies proposed by Landon and Zoe.

**Landon’s Solution**

\[
\frac{4}{5}x = \frac{1}{10}
\]

Scale \( \frac{4}{5} \) up to \( \frac{8}{10} \).

\[
\frac{8}{10}x = \frac{1}{10}
\]

\( 8x = 1 \)

<table>
<thead>
<tr>
<th>1x</th>
<th>1x</th>
<th>1x</th>
<th>1x</th>
<th>1x</th>
<th>1x</th>
<th>1x</th>
<th>1x</th>
</tr>
</thead>
<tbody>
<tr>
<td>8x</td>
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<td></td>
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</tr>
<tr>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td></td>
</tr>
</tbody>
</table>

**Zoe’s Solution**

\[
x
\]

\[
\frac{1}{5}x \quad \frac{1}{5}x \quad \frac{1}{5}x \quad \frac{1}{5}x \quad \frac{1}{5}x
\]

\[
\frac{1}{10} \quad \text{of} \quad \frac{1}{10}
\]

\[
\frac{1}{40} \quad \frac{1}{40} \quad \frac{1}{40} \quad \frac{1}{40} \quad \frac{1}{40}
\]

\[
\frac{5}{40}
\]

2. Explain Landon’s solution strategy.

a. What type of reasoning did Landon use at the beginning of his solution?

b. How did he know to write \( 8x = 1 \)?

c. Will scaling up always work?

3. Explain how Zoe’s solution is similar to the other equations you have solved with bar models.
Use a bar model to solve each equation.

4. \( \frac{3}{4}x = \frac{2}{5} \)

5. \( \frac{2}{7}x = \frac{4}{9} \)

6. You learned to solve addition equations by first reasoning with bar models and then with inverse operations. Now that you have solved multiplication equations by reasoning with bar models, how do you think that you can solve these equations without using the bar models?
Like addition equations, all of the multiplication equations you have modeled in this lesson can be solved with one step. You can use the Properties of Equality and inverse operations to isolate the variable. What operation is the inverse of multiplication?

**WORKED EXAMPLE**

Solve the equation \(4r = 32\).

\[
\begin{align*}
4r &= 32 \\
4(1r) &= 4(8) \\
\frac{4(1r)}{4} &= \frac{4(8)}{4} \\
1r &= 1(8) \\
r &= 8
\end{align*}
\]

- Write equivalent expressions with similar structure.
- Use inverse operations to reverse the multiplication of 4 and 1r.
- Perform division.
- Identity Property of Multiplication

1. Examine the worked example.
   a. Check the solution to \(4r = 32\).

   b. Are there other solutions to the equation? How do you know?

2. Use the same strategy to solve each equation.
   a. \(8a = 72\)  
   b. \(11t = 132\)
When you worked with one-step addition equations, you used the Subtraction Property of Equality to more efficiently solve the problem. Similarly, you can use the Division Property of Equality to solve multiplication problems.

3. Write the properties that justify each step.

\[
\begin{align*}
6w &= 90 \\
\frac{6w}{6} &= \frac{90}{6} \\
1w &= 15 \\
w &= 15
\end{align*}
\]

4. Diego and Venita are solving the equation \(5 = \frac{p}{7}\).

a. Diego says that to solve \(5 = \frac{p}{7}\), he would divide by 7. The value of \(p\) that makes the equation true is \(\frac{5}{7}\). Venita disagrees and says that they should divide by \(\frac{1}{7}\), and the solution is 35. Who is correct?

b. How can Diego and Venita check to see whose answer is correct?
5. Compare the solution strategies used by Sydney and Kailey. What do you notice?

Sydney
\[
\begin{align*}
\frac{2}{5}x &= 20 \\
\frac{2}{5}x \cdot \frac{5}{2} &= 20 \cdot \frac{5}{2} \\
1x &= 20 \left(\frac{5}{2}\right) \\
x &= 50
\end{align*}
\]

Kailey
\[
\begin{align*}
\frac{2}{5}x &= 20 \\
\left(\frac{5}{2}\right) \cdot \frac{2}{5}x &= \left(\frac{5}{2}\right)20 \\
1x &= 50 \\
x &= 50
\end{align*}
\]

6. Solve each equation. Check to ensure that your solution makes the original equation a true statement.

a. \( \frac{n}{4} = 7 \)

b. \( 18 = 3y \)

c. \( \frac{3}{2}h = \frac{5}{2} \)

d. \( 3.14s = 81.2004 \)

e. \( 3\frac{1}{3} = \frac{3}{10}w \)

f. \( 4.2k = 14.7 \)
Recall that an equation is created by writing two expressions with an equals sign between them. Equations can be sometimes, always, or never true.

Consider the equation $7c = 28d$.

1. How is this equation different from the equations you have solved in this lesson?

2. Generate at least 3 pairs of values for $c$ and $d$ that make the equation true.

3. What patterns do you notice?

You can use properties of arithmetic and algebra, along with the properties of equality, to solve for one of the variables in terms of the other variable.

**WORKED EXAMPLE**

\[
\begin{align*}
12a &= 84b \\
\text{Step 1} &\quad 12a = (12 \cdot 7)b \\
\text{Step 2} &\quad 12a = 12(7b) \\
\text{Step 3} &\quad a = 7b
\end{align*}
\]

4. Analyze the worked example.
   a. Why was 84 decomposed into $12 \cdot 7$?
b. What property was applied in Step 2?

c. Explain the reasoning from Step 2 to Step 3. Which property was used?

5. Jesse and Dominic each proposed a solution for the equation \(7c = 28d\). Who’s correct?

<table>
<thead>
<tr>
<th>Jesse</th>
<th>Dominic</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7c = 28d)</td>
<td>(7c = 28d)</td>
</tr>
<tr>
<td>(7c = (7 \cdot 4)d)</td>
<td>((\frac{4}{4})7c = 28d)</td>
</tr>
<tr>
<td>(7c = 7(4d))</td>
<td>((\frac{4 \cdot 7}{4})c = 28d)</td>
</tr>
<tr>
<td>(c = 4d)</td>
<td>(28(\frac{c}{4}) = 28d)</td>
</tr>
<tr>
<td></td>
<td>(\frac{c}{4} = d)</td>
</tr>
</tbody>
</table>

Use reasoning to solve each equation for one of the variables.

6. \(18m = 54n\)  
7. \(12s = \frac{1}{2}t\)
**TALK the TALK**

**What’s Your Strategy?**

Each equation in this lesson is written as \( px = q \), where \( p \) and \( q \) are positive rational numbers and \( x \) is the unknown. You have investigated different strategies to solve these equations.

Analyze each given equation.
- Do you recognize a fact family relationship between the numerical coefficient and the constant?
- Is the numerical coefficient a whole number? A fraction? Or a decimal?
- Do you recognize a way to form a numerical coefficient of 1 using a Property of Equality?

\[
\begin{align*}
2n &= 12 & \frac{2}{5}x &= 14 & 3x &= 55 & 1.1m &= 5.5 \\
1.45r &= 5.9 & 7h &= 35 & \frac{x}{4} &= \frac{3}{8} & 8r &= \frac{3}{4}
\end{align*}
\]

1. Sort each equation according to the solution strategy you think is most efficient.

<table>
<thead>
<tr>
<th>Use Number Sense to Write Equivalent Expressions</th>
<th>Division Property of Equality</th>
<th>Multiplication Property of Equality</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>

2. Provide a rationale for your choice of solution strategy or strategies.
Assignment

Write
Explain how to solve the equation \( px = q \) for \( x \). Be sure to include the properties you use in the process.

Remember
A solution to an equation is the value or values for the variable that makes the equation true. To solve a one-step multiplication equation, isolate the variable using number sense, the Division Property of Equality, or the Multiplication Property of Equality.

Practice
1. Solve each equation using a bar model.
   a. \( 3x = 10 \) 
   b. \( \frac{x}{5} = 6 \)
   c. \( \frac{3}{5} x = 12 \) 
   d. \( \frac{5}{4} x = \frac{2}{3} \)

2. Solve each equation. Check your solutions.
   a. \( 2.1 = 0.5y \) 
   b. \( 4r = 26 \)
   c. \( \frac{2}{9} h = 8 \) 
   d. \( \frac{4}{3} = \frac{8}{3} b \)
   e. \( 14 = \frac{s}{3} \) 
   f. \( 3.8x = 2.736 \)

3. Bertrand invites 21 people to his party and wants to give each guest 3 party favors. If \( n \) is the total number of party favors he will need to order, the equation that represents this situation is \( \frac{n}{21} = 3 \).
   a. If Bertrand orders 58 party favors, will he be able to give each guest 3 party favors? That is, is 58 a solution to the equation?
   b. If Bertrand orders 62 party favors, will he be able to give each guest 3 party favors?
   c. How many party favors does Bertrand need to order? Use the equation to determine the solution. State the inverse operation needed to isolate the variable. Then, solve the equation. Check your solution.

LESSON 3: Play It In Reverse • M3-133
Stretch

Like bar models, balances are also used to model equation solving. Consider the balances shown.

On balance A, a water pitcher balances with a juice bottle. On balance B, the water pitcher balances a cereal bowl and plate. On balance C, three plates balance two juice bottles. How many cereal bowls will balance a water pitcher?

Review

Solve each equation. Check your solutions.

1. \(2.6 + j = 7.1\)
2. \(\frac{21}{5} = b + \frac{3}{4}\)

Rewrite each expression as the product of a constant and a sum of terms.

3. \(2x + 5\)
4. \(\frac{1}{2}x + \frac{3}{5}\)

Determine the conversion.

5. 6 inches = ________ centimeters
6. 10 kilometers = ________ miles
LESSON 4: Getting Real

Solving Equations to Solve Problems

WARM UP

Determine each quotient using long division.

1. \( 435 \div 25 \)
2. \( 511 \div 30 \)
3. \( 860 \div 23 \)

LEARNING GOALS

• Use variables to represent quantities in expressions describing real-world values.
• Solve problems by writing and solving equations.
• Interpret remainders in division problems.

KEY TERM

• literal equation

You know about expressions and equations and how they often represent the structure of real-world situations. How can you apply your knowledge to write equations and solve real-world and mathematical problems?
Equations, Literally

You have already learned a lot of important equations in mathematics. Some of these equations are literal equations. **Literal equations** are equations in which the variables represent specific measures. You most often see literal equations when you study formulas.

For example, the formula for the area of a triangle, \( A = \frac{1}{2} bh \), is a literal equation. The variables in this equation represent the measures of the area, base, and height of the triangle.

1. Consider the formula for area of a parallelogram.
   a. Write the formula for area, \( A \), use \( b \) to represent the base and \( h \) to represent the height.
   b. Solve the equation for \( b \).
   c. Solve the equation for \( h \).

2. The total cost, \( t \), of an online order is the cost of the items, \( c \), plus the cost of shipping, \( s \).
   a. Write an equation to represent the total cost.
   b. Solve the equation for the cost of the items.
   c. Solve the equation for the cost of shipping.

3. You can calculate the distance, \( d \), of an object traveling at a constant rate by multiplying the rate, \( r \), by the time, \( t \). Write an equation in terms of each quantity.
   a. distance        b. rate        c. time
Write and solve an equation for each problem. Show your work and label your answers. Describe the strategy you used to determine each solution.

1. Raul’s sister is 6 years older than he is. What is Raul’s age if Raul’s sister is 19 years old?

2. Approximately \( \frac{1}{10} \) of the mass of a medium-sized apple is sugar. What is the approximate mass of a medium-sized apple that contains 19 grams of sugar?

3. Oscar made brownies for his class. He tripled the recipe he normally uses. If he made 36 brownies for his class, how many brownies does his original recipe make?
4. In June of 2016, for every 20 total emails a person received, they could expect to get 11 spam emails. If a person received 300 spam emails in one month, how many total emails did they receive?

5. In Jaden’s town, the middle school has 443 more students than the high school. If the middle school has 817 students, how many students are at the high school?

6. The average height of an ostrich, the tallest bird, is 121 inches. The average height of a bee hummingbird, the smallest bird, is 2.75 inches. How many times taller is the ostrich than the bee hummingbird?
For each question, write an equation to represent the situation and then solve it to answer the equation. A situation may require more than one equation.

1. Kendra bought some back-to-school supplies for $1.70. She showed them to her friend Naya: 2 erasers for 2 cents each, 5 markers for 4 cents each. She also bought 8 notepads, but she forgot how much she paid for them. She did not pay sales tax.

Naya said that she was not charged the right amount. How did she know?

2. The Bermuda Triangle is an imaginary triangle connecting Miami, Florida, to San Juan, Puerto Rico, to Bermuda. The Bermuda Triangle covers an area of 454,000 square miles. The dashed line on the map shows a distance of about 926 miles.

What is the approximate distance from Bermuda to Puerto Rico?

3. Amit’s school is an unusual school. It has 1677 students in grades 3–6. There are twice as many fifth graders as sixth graders and three times as many fourth graders as fifth graders. Finally, there are five times as many third graders as fourth graders. How many fifth graders are in Amit’s school?
4. There are two routes Jasmine can take when she bikes home from school—the long way and the short way. The long way is $1\frac{1}{2}$ times as far as the short way. During one week, she biked a total of 30 miles from school to home. She took the short way three times.

a. What is the distance of the short way?

b. What is the distance of the long way?

---

**Activity 4.3**

Interpreting Remainders in Solutions

1. The Red Cross disaster relief fund collected 3551 winter coats to distribute to flood victims. If there are 23 distribution centers, how many coats can be sent to each center? Marla’s calculations are shown.

Marla said, “The Red Cross can send $154\frac{9}{23}$ coats to each center.” Madison replied, “You cannot have a fraction of a coat. So, each center will receive 154 coats and there will be 9 coats left over.”

Who’s correct and why?
In division problems, the remainder can mean different things in different situations. Sometimes the remainder can be ignored, and sometimes the remainder is the answer to the problem. Sometimes the answer is the number without the remainder, and sometimes you need to use the next whole number up from the correct answer.

2. The Carnegie Middle School is hosting a picnic for any fifth grader who will be attending school next year as a sixth grader. The hospitality committee is planning the picnic for 125 students. Each fifth grader will get a sandwich, a drink, and a dessert.

   a. The hospitality committee is ordering large sandwiches that each serve 8 people. If 125 fifth graders are coming to the picnic, how many sandwiches should the committee buy?

   b. The committee is planning to have frozen fruit bars for dessert. If frozen fruit bars come in boxes of 12, how many boxes of frozen fruit bars should they order?

   c. They will be serving bottles of water. Bottled water comes in cases of 24. How many cases of water will they need? Will there be any extra bottles of water? If so, how many?

   d. The fifth graders will take a bus from the elementary school to the middle school on the afternoon of the picnic. If each bus seats 32 passengers, how many buses will be needed to transport the students? How many seats will be empty?

In other words, you can round down if you don't need to use the remainder, and you can round up if you need the next whole number larger than your answer.
3. Throughout the year, local businesses collected 28,654 pairs of eyeglasses for disaster victims. If they have requests from 236 relief organizations, how many pairs of eyeglasses can each organization receive? How many pairs, if any, will be left over?

**TALK the TALK**

**Write Your Own**

1. Write your own word problem that can be solved by writing and solving the equation $2.4 + x = 5$.

2. Write your own word problem that can be solved by writing and solving the equation $4x = 8$. 
**Practice**

1. Solve each equation. Show your work.
   a. $3y = 18$
   b. $m + 12 = 29$
   c. $3g = 6.3$
   d. $5x = 12 + 18$
   e. $2(a + 2a) = 90$

For each problem, write an equation to represent the situation and then solve it to answer the question.

2. A rectangular pool has a width of 24 feet. A second rectangular pool has a perimeter of 48 feet, which is $\frac{1}{3}$ the perimeter of the first pool.

3. The local firefighters collect toys to distribute at various give-away events. They have 4569 toys and will sponsor 129 give-away events. How many toys can they give away at each event? How many toys, if any, will be left over?

**Stretch**

You read a report that says that only $\frac{7}{100}$ of all people who own car dealerships in the country are women.

1. There are about 20,000 people who own car dealerships in the country. How many of them are female?
2. In a group of 2000 people who own car dealerships attending a conference, about how many would you expect to be female?
3. How did you determine the number of women car dealers, given the total number of car dealers? Use complete sentences to explain your answer.
4. Write an expression to represent the number of women car dealers, given the total number of car dealers.
5. Write an equation that you can use to determine the total number of car dealers in a certain city, given that the number of women car dealers in the city is 14.
6. Use the equation to determine the total number of car dealers in the city.
Review

1. Rewrite each algebraic expression by applying the Distributive Property.
   a. 3.5(2x + 1)
   b. $\frac{1}{2}(3 - 4a)$

2. Create a bar model to solve each equation.
   a. 4x = 30
   b. $\frac{2}{3}x = 21$

3. Write a different division problem that has the same quotient as the one given. Explain your answer.
   a. 36.5 ÷ 0.005  
   b. 63.196 ÷ 14.8
An equation is a statement of equality between two expressions. An equation can contain numbers, variables, or both in the same mathematical sentence. Equations may be always true, never true, or true only for one or more values of the variable. The Reflexive Property of Equality says that when both sides of an equation look exactly the same, their values are equal.

A solution to an equation is any value for the variable that makes the equation true.

Properties of Equality are logical rules that allow you to maintain balance and rewrite equations.

<table>
<thead>
<tr>
<th>Always True</th>
<th>Never True</th>
<th>True for certain values of the variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6 = 10 - 4$</td>
<td>$10 = 20$</td>
<td>$x = 5$</td>
</tr>
<tr>
<td>$x = x$</td>
<td>$x = x + 2$</td>
<td>$x + 2 = 12$</td>
</tr>
</tbody>
</table>
Properties of Equality

<table>
<thead>
<tr>
<th>Properties of Equality</th>
<th>For all numbers $a$, $b$, and $c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Addition Property of Equality</td>
<td>If $a = b$, then $a + c = b + c$.</td>
</tr>
<tr>
<td>Subtraction Property of Equality</td>
<td>If $a = b$, then $a - c = b - c$.</td>
</tr>
<tr>
<td>Multiplication Property of Equality</td>
<td>If $a = b$, then $a \cdot c = b \cdot c$.</td>
</tr>
<tr>
<td>Division Property of Equality</td>
<td>If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.</td>
</tr>
<tr>
<td>Symmetric Property of Equality</td>
<td>If $a = b$, then $b = a$.</td>
</tr>
</tbody>
</table>

Equations that have an infinite number of solutions are equations that are true regardless of the value you assign to the variable. These kinds of equations often describe important properties of numbers. For example:

- The **Zero Property of Multiplication** states that the product of any number and 0 is 0: $x \cdot 0 = 0$.
- The **Identity Property of Multiplication** states that the product of any number and 1 is the number: $x \cdot 1 = x$.
- The **Identity Property of Addition** states that the sum of any number and 0 is the number: $x + 0 = x$.

You can use a number line to represent inequalities. The **graph of an inequality** in one variable is the set of all points on a number line that make the inequality true. The set of all points that make an inequality true is the **solution set of the inequality**.

The solution to any inequality can be represented on a number line by a ray. A ray begins at a starting point and goes on forever in one direction. A closed circle means that the starting point is part of the solution set of the inequality. An open circle means that the starting point is not part of the solution set of the inequality.

For example, the solution set of the inequality $x \leq 3$ is all numbers equal to or less than 3, and the solution set of the inequality $x < 3$ is all numbers less than 3.
A bar model uses rectangular bars to represent known and unknown quantities.

For example, the equation \( x + 10 = 15 \) states that for some value of \( x \), the expression \( x + 10 \) is equal to 15. This can be represented using a bar model.

The expression \( x + 10 \) can be decomposed into a part representing \( x \) and a part representing 10. The number 15 can be decomposed in a similar way: \( 15 = 5 + 10 \). The bar model demonstrates that these two equations are equivalent:

\[
\begin{align*}
    x + 10 & = 15 \\
    x + 10 & = 5 + 10
\end{align*}
\]

By examining the structure of the second equation, you can see that 5 is the value for \( x \) that makes this equation true.

A one-step equation is an equation that can be solved using only one operation. To solve a one-step addition equation, isolate the variable using number sense or inverse operations. Inverse operations are pairs of operations that reverse the effects of each other.

For example, solve the equation \( h + 6 = 19 \).

\[
\begin{align*}
    h + 6 & = 13 + 6 & \text{Write equivalent expressions that mirror structure.} \\
    h + 6 - 6 & = 13 + 6 - 6 & \text{Use inverse operations to reverse the addition of 6 to } h. \\
    h + 0 & = 13 + 0 & \text{Combine like terms and apply the Additive Identity Property.} \\
    h & = 13
\end{align*}
\]
You can also use bar models to reason about the solution to multiplication equations.

For example, the equation $2x = 6$ states that for some value of $x$, the expression $2x$ is equal to 6. You can decompose $2x$ by rewriting it as the equivalent expression $x + x$, or $x + x$. To maintain equivalence, decompose 6 in a similar way. The bar model demonstrates that these two equations are equivalent.

\[
\begin{align*}
2x &= 6 \\
x + x &= 3 + 3
\end{align*}
\]

By examining the structure of the second equation, you can see that $x = 3$.

Bar models can also be used to solve multiplication equations with fractional coefficients.

For example, represent $\frac{1}{3}x = 2$ as a bar model.

To solve this equation for $x$, compose 3 equally-sized parts to create the whole, $x$. To maintain equivalence, compose 3 equally-sized parts for the other expression too. This structure allows you to see the value of $x$ that makes the equation true: $x = 6$.

You can also use the inverse operation of multiplication to solve one-step multiplication equations.

For example, solve the equation $4r = 32$.

\[
\begin{align*}
4r &= 32 \\
4(1r) &= 4(8) \quad \text{Write equivalent expressions with similar structure.} \\
\frac{4(1r)}{4} &= \frac{4(8)}{4} \quad \text{Use inverse operations to reverse the multiplication of 4 and 1r.} \\
1r &= 1(8) \quad \text{Perform division.} \\
r &= 8 \quad \text{Identity Property of Multiplication}
\end{align*}
\]
You can use properties of arithmetic and algebra, along with the properties of equality, to solve for one of the variables in an equation in terms of the other variable.

\[ 12a = 84b \]

**Step 1**  
\[ 12a = (12 \cdot 7)b \]

**Step 2**  
\[ 12a = 12(7b) \]

**Step 3**  
\[ a = 7b \]

**Getting Real**

**Literal equations** are equations in which the variables represent specific measures. You most often see literal equations when you study formulas. The formula for the area of a triangle, \[ A = \frac{1}{2}bh, \] is a literal equation. The variables represent the measures of the base and height of the triangle.

In division problems, the remainder can mean different things in different situations. Sometimes the remainder can be ignored, and sometimes the remainder is the answer to the problem. Sometimes the answer is the quotient without the remainder, and sometimes you need to use the next whole number up from the quotient.

For example, the Red Cross disaster relief fund collected 4233 winter coats to distribute to flood victims. If there are 28 distribution centers, how many coats can be sent to each center?

\[ 4233 \div 28 = 151 \frac{5}{28} \]

You cannot have a fraction of a coat, so each center will receive 151 coats and there will be 5 coats left over.
On a long run, runners keep track of their splits. For example, on a half-marathon (13.1-mile run), the runner’s time is measured at 5 miles, 10 miles, and at the finish line.

Lesson 1
Every Graph Tells a Story
Independent and Dependent Variables

Lesson 2
The Power of the Horizontal Line
Using Graphs to Solve One-Step Equations

Lesson 3
Planes, Trains, and Paychecks
Multiple Representations of Equations

Lesson 4
Triathlon Training
Relating Distance, Rate, and Time
TOPIC 3: GRAPHING QUANTITATIVE RELATIONSHIPS

Students learn that quantities can vary in relation to each other and are often classified as independent and dependent quantities. They solve for unknown values of a variable by analyzing a graph. Students then solve linear equations using the variety of tools available to them, and they contrast the advantages and limitations of each. Throughout the topic, students compare and contrast linear equations of the form \( y = x + c \) and \( y = cx \), and their respective representations. Finally, they analyze three different distance, rate, and time scenarios and generalize the formula \( d = rt \). Students are expected to use their knowledge of unit rate and unit conversion to solve these problems.

Where have we been?
In grade 5 as well as in previous topics, students graphed and named ordered pairs in the first quadrant of the coordinate plane. This topic begins by asking students to interpret graphs located only in Quadrant I. It also connects to graphs of equivalent ratios and writing inequalities as constraints, which students learned about in previous lessons.

Where are we going?
This topic provides the foundation for the formal study of independent and dependent variables, which will be revisited later in this course. In Module 4 and then in grade 7, students will continue analyzing different representations of scenarios, but they will no longer be restricted to Quadrant I. This topic also combats a common misconception – that all relationships between variables are linear – by exposing students to nonlinear graphs and scenarios.

Using a Graph to Visualize an Equation

In this example, the graph models the expression \( 1.25x \), where \( x \) is the number of pretzels sold, and \( 1.25x \) is the amount of money collected. This graph intersects a horizontal line which represents the expression 10. The \( x \)-coordinate of the point at which the lines intersect represents the solution to the equation 10 = 1.25x, so \( x = 8 \).
Myth: Memory is like an audio or video recording.

Let’s play a game. Memorize the following list of words: strawberry, grape, watermelon, banana, orange, peach, cherry, blueberry, raspberry. Got it? Good. Some believe that the brain stores memories in pristine form. The idea is that memories last for a long time and don’t change – like recordings. Without looking back at the original list, was apple on it?

If you answered “yes,” then go back and look at the list. You’ll see that apple does not appear, even though it seems like it should. In other words, memory is an active, reconstructive process that takes additional information, like the category of words (e.g., fruit), and makes assumptions about the stored information.

This simple demonstration suggests memory is not like a recording. Instead, it is influenced by prior knowledge and decays over time. Therefore, students need to see and engage with the same information multiple times to minimize forgetting (and distortions).

#mathmythbusted

Talking Points

You can further support your student’s learning by asking questions about the work they do in class or at home. Your student is learning to make connections between algebraic representations and graphical representations of mathematical objects.

Some Things to Look For

Discuss graphs you see online, on television, and in print. Talk about what the graph is demonstrating and what two (or more) quantities it is comparing.

Key Terms

discrete graph
A discrete graph is a graph of isolated points.

continuous graph
A continuous graph is a graph with no breaks in it; each point on it represents a solution to the graphed scenario.

dependent quantity
When one quantity depends on another in a problem situation, it is said to be the dependent quantity.

independent quantity
The quantity on which the dependent quantity depends is called the independent quantity.
LESSON 1: Every Graph Tells a Story

Independent and Dependent Variables

WARM UP
Write an inequality for each verbal statement.

1. $x$ is less than 5.

2. 4 times $g$ is no more than 9.

3. $y$ is at least 2 more than $x$.

4. 3 less than the product of 4 and some number is greater than another number.

LEARNING GOALS
- Interpret information about a situation from a graphical representation.
- Determine whether graphs are discrete or continuous.
- Identify the graphs of situations.
- Identify and use variables to define independent and dependent quantities in real-world problems.
- Write an equation to express a quantity that is the dependent variable in terms of another quantity, the independent variable.

KEY TERMS
- discrete graph
- continuous graph
- dependent quantity
- independent quantity
- independent variable
- dependent variable

Throughout this course, you have analyzed quantities in a variety of ways. Often, the equation you write to represent variable quantities depends on the question you are answering. How do you tell what variable quantity is the focus of a mathematical question?
Getting Started

It’s Not a Tall Tale!

Write a story to describe the situation represented by each graph.

1. The Water Level in the Bathtub

2. Money in Your Bank Account

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In this activity, you will match a specific graph to a real-world problem situation.

Cut out the graphs and scenarios located at the end of the lesson.

1. **Tape each graph in the box with the appropriate scenario.**
   Label the axes with appropriate quantities and units.

2. **How did you determine which graph matched which scenario?**

   Even though only certain points make sense in the situation, you can draw a line to represent the shape of discrete data.
A **discrete graph** is a graph of isolated points. Often, the coordinates of those points are counting numbers. A **continuous graph** is a graph with no breaks in it. Each point on a continuous graph, even those with fractional numbers as coordinates, represents a solution to the graphed scenario.

3. Which graphs are discrete graphs and which are continuous graphs? How does the scenario inform you that the graph will display discrete points or be continuous?

4. Which graph(s) represent equivalent ratios? How does the scenario inform you that the graph(s) will display equivalent ratios?

5. Consider the graph in the Rainy Day scenario. Assume that 2 hours after you left home, 1.5 inches of rain had fallen.
   a. Explain how the graph illustrates that the rain fell faster later in the day than at the beginning of the day.
   b. Write an inequality statement in terms of the time, \( t \), to represent when the rain stopped for the day.
6. Consider the graph in the Toy Rocket scenario. The rocket reaches a maximum height of 256 feet.
   a. Describe the shape of the graph.

   b. Write an inequality statement in terms of the time, \( t \), to represent when the rocket was rising into the air.

7. Consider the graph of the T-shirt Sales scenario. Suppose there is a minimum order total of $100 when you are ordering the T-shirts. Write an inequality statement for the number of shirts, \( n \), that must be ordered to meet the minimum order requirement.

Be sure to keep your graphs and scenarios. You will use them in the next activity.
When one quantity depends on another in a real-world problem situation, it is said to be the **dependent quantity**. The quantity on which it depends is called the **independent quantity**. The variable that represents the independent quantity is called the **independent variable**, and the variable that represents the dependent quantity is called the **dependent variable**.

Consider the scenarios from the previous activity.

**1. Use the Pool Party scenario to answer each question.**
   
   a. What two quantities are changing in this situation?
   
   b. Which quantity depends on the other?
   
   c. Define variables for each quantity and label them appropriately as the independent and dependent variables.

**2. Use the Fish Tank scenario to answer each question.**
   
   a. What two quantities are changing in this situation?
   
   b. Which quantity is the independent quantity and which is the dependent quantity?
c. The equation that represents the fish scenario is \( w = 200 - 10t \). What do the variables \( w \) and \( t \) represent in this equation?

d. What do you notice about which variable is isolated in the equation?

3. Identify the independent quantity and the dependent quantity in each of the four remaining scenarios.

a. Rainy Day
   
   **Independent Quantity:**

   **Dependent Quantity:**

b. Toy Rocket
   
   **Independent Quantity:**

   **Dependent Quantity:**

c. DVD and Game Rentals
   
   **Independent Quantity:**

   **Dependent Quantity:**

d. T-shirt Sales
   
   **Independent Quantity:**

   **Dependent Quantity:**

Examine the graphs. Do you see any connection between the independent and dependent variables and the graph?
A store makes 20% profit on the total price of all the items they sell.

Analyze the situation.

1. **Name the two quantities that are changing.**

2. **Describe which value depends on the other.**

Let $t$ represent the total price of all items sold in dollars, and let $p$ represent the profit in dollars.

3. **Write an equation to represent the relationship between these variables.**

4. **Identify the independent and dependent variables in this situation.**
5. Complete the table.

<table>
<thead>
<tr>
<th>Quantity Name</th>
<th>Unit of Measure</th>
<th>Independent Quantity</th>
<th>Dependent Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>25.00</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>49.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>99.95</td>
<td></td>
</tr>
</tbody>
</table>

6. Use the table to complete the graph.

7. Is this a discrete graph or a continuous graph? Explain.

8. On which axis is the independent variable? On which axis is the dependent variable?
Let's think about the problem situation in a different way.

Suppose you are operating this store and you know how much profit you make on each item.

1. Name the two quantities that are changing.

2. Describe which value depends on the other.

Let $p$ be equal to the profit, and let $t$ be equal to the total price of all items sold.

3. Write an equation to represent the relationship between these variables.

4. Identify the independent and dependent variables in this situation.
5. Complete the table.

<table>
<thead>
<tr>
<th>Independent Quantity</th>
<th>Dependent Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity Name</td>
<td></td>
</tr>
<tr>
<td>Unit of Measure</td>
<td></td>
</tr>
<tr>
<td>Variable</td>
<td></td>
</tr>
<tr>
<td>7.50</td>
<td></td>
</tr>
<tr>
<td>10.00</td>
<td></td>
</tr>
<tr>
<td>19.99</td>
<td></td>
</tr>
</tbody>
</table>

6. Use the table to complete the graph.

7. Is this a discrete graph or a continuous graph? Explain.

8. On which axis is the independent variable? On which axis is the dependent variable?
The situations in the previous activities, *Total Price and Profit* and *Profit and Total Price*, are similar but presented in two different ways.

1. **Complete each summary statement.**

<table>
<thead>
<tr>
<th>Total Price and Profit</th>
<th>Profit and Total Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>The __________ depends on the __________.</td>
<td>The __________ depends on the __________.</td>
</tr>
<tr>
<td>Equation:</td>
<td>Equation:</td>
</tr>
</tbody>
</table>

2. **What do you notice about the two equations?**

3. **How does examining this same situation from different perspectives affect the independent and dependent variables?**

4. **What can you conclude about the designation of a variable as independent or dependent?**
5. Compare the two graphs in the activities Total Price and Profit and Profit and Total Price.
   a. How are they similar and how are they different?
   
   b. What do you notice about the independent and dependent variables?

Consider another scenario.

Dawson purchased a diesel-powered car that averages 41 miles per gallon.

6. Suppose Dawson is interested in how far the car travels on a given amount of gas.
   a. Identify the independent and dependent quantities.
   
   b. Define variables for each quantity and identify which is the independent variable and which is the dependent variable.
   
   c. Write an equation to represent the relationship between the two variables.
7. Suppose, instead, that Dawson runs out of gas on a regular basis. He is interested in how many gallons of gas he has used if he knows how many miles he has driven. Use the same variables you defined in Question 6.

   a. Identify which variable represents the independent quantity and which variable represents the dependent quantity.

   b. Write an equation to represent the relationship between the two variables.

8. How would you expect the graphs of the two situations to be similar? How would they be different?
TALK the TALK

Create Your Own Story

1. Create a real-world situation to match the given graph.

   ![Graph](image)

2. Identify the independent and dependent quantities in your scenario.
3. Label the x-axis and y-axis to reflect your scenario.

4. What real-world question could be answered using the graph based on your scenario.
 Lesson 1: Every Graph Tells a Story  •  M3-171
Pool Party

You have cookies for your team pool party, but you don't know how many of your teammates will show up. How many cookies will each teammate receive if everyone receives the same number of cookies?
Fish Tank

You are draining a 200-gallon fish tank at a rate of 10 gallons per minute. How much water remains in the tank at a specific time?

Rainy Day

When you left home, the rain was falling at a steady rate. Then, it stopped raining for a few hours before a sudden downpour. Finally, it stopped raining. How many inches of rain had fallen at different points of the day?
Toy Rocket

You launch a toy rocket into the air from the ground and observe its height through its entire flight. How many feet high was the rocket at a specific time after launch?

DVD and Game Rentals

The video kiosk charges $2.00 for DVD and game rentals. How many DVDs and games can you rent for different amounts of money?
T-shirt Sales

You buy T-shirts to sell for your school. There is a $25 design charge plus the cost per T-shirt. What is the total cost for different numbers of T-shirts?
Write
Write the term that best completes each statement.
1. In a real-world problem situation, when a quantity does not depend on another quantity, it is called the ____________________.
   In the equation that models the problem situation, this quantity is represented by the ____________________.
2. In a real-world problem situation, when a quantity depends on another quantity, it is called the ____________________.
   In the equation that models the problem situation, this quantity is represented by the ____________________.

Practice
1. Determine the independent variable and the dependent variable in each given equation.
   a. The equation $T = 75 - d$ is used to calculate the water temperature, $T$, at a depth, $d$, in a particular lake.
   b. The equation $p = \frac{t}{3}$ is used to calculate the individual profit, $p$, made by each of three brothers operating a lemonade stand with a total profit, $t$.
2. An online ticket broker charges a flat service fee of $6.50 per ticket sold. You are interested in the total amount of money you must pay for a given number of tickets.
   a. Name the two quantities that are changing in this situation.
   b. Define variables for each quantity and identify which represents the independent quantity and which represents the dependent quantity.
   c. Write an equation for the relationship between the two variables.
3. Jana is a runner. When she is training for a race, she averages 8 miles per hour. She is interested in how far she can run in a given number of hours.
   a. Define variables for each changing quantity and identify each as the independent or dependent variable.
   b. Write an equation to represent this situation.
   c. Use your equation to create a table of values for this situation.
   d. Use your equation and table to create a graph. Remember to label your axes.
   e. Explain how you knew which variable to graph on each axis.
   f. Rewrite the equation from part (b) with the other variable isolated.
   g. With the equation in this form, which variable is the independent and which is the dependent? Explain your reasoning.
   h. Write a question for which the equation in (f) would be needed.
**Stretch**
Create two different scenarios that use time as a varying quantity. In one scenario, use time as the independent quantity. In the other, use time as the dependent quantity. Write a question that could be answered in each case. Create a graph for each situation.

**Review**
1. When Sarah goes out to eat, she always tips her server 18% of the bill. She also must pay 7% sales tax on her dinner.
   a. Define variables for the quantities in the situation.
   b. Write an equation for the total cost of Sarah’s meal, including tax and tip.
   c. Suppose Sarah paid a total of $31.25. How much was her meal?

2. A builder requires a certain number of bricks each time he builds a brick structure. To make sure he has enough bricks, he always orders 50 additional bricks.
   a. Define variables for the quantities in the situation.
   b. Write an equation for the total number of bricks ordered.
   c. Suppose the builder calculated that the needed 1275 bricks. How many bricks were ordered?

3. Solve each equation and state the inverse operation you used.
   a. \( t + \frac{3}{4} = 8 \)
   b. \( 22 = \frac{11}{7} y \)

4. Write the two possible unit rates for each ratio.
   a. 8 cups of sugar for every 2 tablespoons of vanilla
   b. $3.56 for 24 ounces
LEARNING GOALS

- Analyze the relationship between the independent and dependent variables in a graph and relate the variables to an equation.
- Use multiple representations to solve one-step real-world problems.
- Use an inequality of the form $x > c$ or $x < c$ to represent constraints when solving a real-world problem.

WARM UP

Which equations represent proportional relationships? Explain how you know.

1. $c = 2.5n$
2. $l = w + 25$
3. $d = \frac{1}{3}t$
4. $T = 100 - d$

You have learned how to solve one-step equations using reasoning and the Properties of Equality. How can you use graphs to solve one-step real-world problems?
Selling Pretzels

Nic sells pretzels for $1.25 each at the morning baseball and softball games held at the Community Center. At the end of the games he is supposed to report the number of pretzels he sold and the total amount of money collected.

Nic sold pretzels on three different mornings, but he only reported either the number of pretzels sold or the dollar amount collected.

1. Calculate each missing piece of information from his daily reports.
   
   a. 16 pretzels sold
   
   b. 40 pretzels sold
   
   c. $40 collected

2. Write an equation to represent the relationship between the number of pretzels sold, $x$, and the amount of money collected in dollars, $y$.

3. Does this situation represent a proportional relationship? Justify your answer.
In the previous activity, you answered questions and wrote the equation $y = 1.25x$, where $x$ represents the number of pretzels sold and $y$ represents the amount of money collected in dollars.

The amount of money collected for the number of pretzels sold can be represented by points on the graph. The graph shows the ordered pairs corresponding to the three questions you answered about Nic selling pretzels. The equation corresponding to the graph is $y = 1.25x$.

1. Label the three ordered pairs shown on the graph.

2. What does each ordered pair represent?

3. Identify the unit rate in this situation. Plot and label it on the graph.
You can use a graph to determine an independent quantity given a dependent quantity.

**WORKED EXAMPLE**

You can use the graph to determine how many pretzels Nic sold if he collected $10.

First, locate 10 on the y-axis and draw a horizontal line. This shows that $10 is the amount of money collected. The x-value of the point where your horizontal line intersects with the graph of $1.25x$ is the number of pretzels sold for $10.

4. How many pretzels did Nic sell if he collected $10?
Use the graph in the worked example to answer each question.

5. How many pretzels did Nic sell if he collected:
   a. $25.00?

   b. $33.75?

   c. $75.00?

   d. more than $45.00?

   e. at least $100.00?

6. Nic reported that on Saturday morning he sold 13 pretzels and collected $16.25, and on Saturday afternoon he sold 42 pretzels and collected $55.00.

   Do you think he reported accurately? Explain your reasoning.
An online site sells a single closeout item each day. The items and prices change daily. The company charges a flat fee of $6.00 for shipping.

The graph shown represents a model of this problem situation.

1. Write an equation to represent the relationship between the cost of an item, $x$, and the total cost, $y$. Label the graph of the line with your equation.

2. Does this situation represent a proportional relationship? Justify your answer.
Use the graph to answer each question.

3. What is the total cost if the cost of the item is:
   a. $18.00?     b. $25.00?
   c. $32.50?     d. $75.00?

4. Jeff was considering buying one of the daily closeout items that costs $23.99. He has a $25.00 gift card. Can he afford the total cost of the daily closeout item?

5. Suppose the flat fee for shipping changed to $6.80. How would the graph change? How would the equation change? Would that change the way you could use the graph to determine values?

6. How would the graph change if there was free shipping on all orders where the cost of the item is less than $20.00? Sketch the graph.
TALK the TALK

Plus or Times?

1. Describe the similarities and differences between the pairs of equations.
   a. \( y = 5x \) and \( y = x + 5 \)
   b. \( y = \frac{1}{2}x \) and \( y = \frac{1}{2} + x \)
   c. \( y = 4.95x \) and \( y = x + 4.95 \)

2. Describe the similarities and differences among the graphs shown. Write an equation for each graph.

   Graph A
   Graph B

   Graph C
   Graph D

3. Describe how you can use a graph to solve one-step equations.
Assignment

Write
Describe the similarities and differences between the graphs of equations represented in the form \( y = nx \) and \( y = x + n \), where \( n \) is any positive rational number and \( x \) and \( y \) are unknown quantities.

Remember
Graphs are powerful visual representations of how quantities are related. You can use a graph to estimate a solution. You can also formally solve equations to determine exact values.

Practice
A shuttle space suit, including the life support system, weighs about 310 pounds. The break in the \( y \)-axis represents the values from 0–300.
1. What does each ordered pair on the line represent?
2. Write an equation to represent the relationship shown in the graph.
3. Is this a proportional relationship? Justify your answer.
4. In this problem situation, do all the points on the line make sense? Explain your reasoning.
5. Determine the weight of an astronaut without the shuttle suit given that the astronaut’s weight while wearing the shuttle suit.
   a. 480 lb
   b. 467 lb
   c. 520 lb

The gravitational pull of the Moon is not as great as that on Earth. In fact, if a person checks their weight on the Moon, it will be only \( \frac{1}{6} \) of their weight on Earth.
6. What does each ordered pair on the line represent?
7. Write an equation to represent the relationship shown in the graph.
8. Is this a proportional relationship? Justify your answer.
9. In this problem situation, do all the points on the graph make sense?
10. Determine the weight of a person on Earth given his weight on the Moon.
    a. 12 lb
    b. 21 lb
    c. 36 lb
**Review**
Solve for each unknown value.
1. The area of a triangle is 12.5 square feet and the height is 6 feet. Determine the base of the triangle.
2. The area of a parallelogram is 74.8 square feet and the base is 22 feet. Determine the height of the parallelogram.

Determine the independent and dependent quantities in each scenario.
3. Selena is driving to her grandmother’s house. She travels an average of 60 miles per hour.
4. On her way to work each morning, Sophia purchases a cup of coffee for each of her colleagues and pays $2.25 per cup the coffee shop.

Use long division to determine each quotient.
5. $1968 \div 12$
6. $2363 \div 139$

---

**Stretch**
Write an equation to represent each of the three segments of the graph shown. List any restrictions in the possible x-values.
LESSON 3: Planes, Trains, and Paychecks

Multiple Representations of Equations

WARM UP
Identify two quantities in each situation. Then state which quantity depends on the other.

1. Snow is falling at a rate of 3 inches per hour.
2. The outside temperature is increasing at an average of 10 degrees each day.
3. The car wash generated an average profit of $8.00 per car.
4. The income from the sale of movie tickets was $8.50 per person.
5. A dog groomer charges $35 for every dog.

LEARNING GOALS
- Write and solve equations that represent relationships given in tables, graphs, and situations.
- Identify independent and dependent quantities represented in tables, graphs, and scenarios.
- Analyze the relationship between the independent and dependent quantities in a situation using graphs, tables, and equations.
- Determine whether the data represented in a graph of an equation are discrete or continuous.

You have identified independent and dependent quantities in relationships and have expressed these relationships using equations. How can you relate independent and dependent quantities in a variety of different situations, using a variety of different representations?
Getting Started

To the Equation-Mobile!

A mobile (MO-beel) is hanging art. It features all kinds of different objects suspended from string or wire. Balance is important to the visual effect of mobiles.

Determine what value each shape represents in each mobile.

1.  

2. 
SnapSmart charges the same price for each 3 in. by 4 in. picture print.

1. The table shows a few orders and the cost of each.
   a. What is the cost of one print? Explain how you determined the cost.

   b. Define variables for the quantities in this situation. Then write an equation that models the relationship between these quantities.

   c. Create a graph for this situation.

   d. Tell whether the quantities in the SnapSmart scenario are discrete or continuous. Explain your reasoning.

   e. You can draw a line to show the shape of the graph. Do all the points on the line make sense in this scenario?
2. The table shows the cost of a particular item.

<table>
<thead>
<tr>
<th>Number of Items</th>
<th>Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>48</td>
</tr>
<tr>
<td>16</td>
<td>96</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
</tr>
</tbody>
</table>

a. Describe how the cost is related to the number of items.

b. Define variables for the quantities in this situation. Then write an equation that models the relationship between these quantities.

c. Explain whether the quantities in this situation are discrete or continuous.

d. Create a graph for this situation.

e. What do you notice about the shape of the graph? You can connect the points to see the shape.
3. Read each situation and analyze the corresponding table of values. Identify the independent and dependent quantities in each. Then, write an equation that models the relationship between the quantities.

a. The total profit made on cutting lawns and the profit made by each person are represented in the table shown.

<table>
<thead>
<tr>
<th>Total Profit Made ($)</th>
<th>Profit Made by Each Person ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>7.00</td>
</tr>
<tr>
<td>25.50</td>
<td>8.50</td>
</tr>
<tr>
<td>45</td>
<td>15.00</td>
</tr>
</tbody>
</table>

b. The number of boxes of cookies sold and the total profit are represented in the table shown.

<table>
<thead>
<tr>
<th>Boxes of Cookies Sold</th>
<th>Total Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>7.50</td>
</tr>
<tr>
<td>5</td>
<td>12.50</td>
</tr>
<tr>
<td>7</td>
<td>17.50</td>
</tr>
</tbody>
</table>

A table of values can be represented vertically or horizontally.

c. The number of tiles required to complete a job and the number of tiles ordered are represented in the table shown.

<table>
<thead>
<tr>
<th>Number of Tiles Required</th>
<th>75</th>
<th>95</th>
<th>115</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Tiles Ordered</td>
<td>90</td>
<td>110</td>
<td>130</td>
</tr>
</tbody>
</table>
The graph shows the relationship between the distance of a train from the station and the time in minutes.

1. Complete the table using the points from the graph.

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>Distance from Station (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

2. Define variables and write an equation to represent the relationship between the quantities.

3. How far is the train from the station after 20 minutes?

4. Connect the points to show the shape of the graph. Do all of the values on the line make sense in this situation?
The graph shows the relationship between the height of a plane and its distance from the airport in miles.

5. Complete the table using the points from the graph.

<table>
<thead>
<tr>
<th>Distance (miles)</th>
<th>Height (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>50</td>
<td>10,000</td>
</tr>
<tr>
<td>100</td>
<td>20,000</td>
</tr>
<tr>
<td>200</td>
<td>20,000</td>
</tr>
</tbody>
</table>

6. Write an equation to represent the relationship between the quantities.

7. What is the height of the plane at 150 miles?

8. Connect the points to show the shape of the graph. Do all of the values on the line make sense in this situation?
Crystal got a job working at the local hardware store making $8.76 per hour.

1. Write an equation that models the relationship between the number of hours Crystal worked and how much she earned.

2. How much would Crystal earn if she worked the given times.
   a. 5 hours
   b. $2\frac{1}{2}$ hours
   c. 5 hours and 30 minutes
   d. 10 hours and 15 minutes

3. Use your equation to calculate the number of hours Crystal worked given her total pay.
   a. $218.75
   b. $293.46
   c. $203.67
4. Complete the table.

<table>
<thead>
<tr>
<th>Time Worked (hours)</th>
<th>Earnings ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td></td>
</tr>
<tr>
<td>5.5</td>
<td></td>
</tr>
<tr>
<td>10.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>218.75</td>
</tr>
<tr>
<td></td>
<td>293.46</td>
</tr>
<tr>
<td></td>
<td>203.67</td>
</tr>
</tbody>
</table>

5. Use the table to complete the graph.

6. Connect the points to show the shape of the graph. Do all of the points on the line make sense in this situation?
7. Jake’s dog eats an average of 40 pounds of dry dog food in one month.

   a. Write an equation to model the relationship between the number of pounds of dog food and the number of months.

b. Complete the table and graph.

<table>
<thead>
<tr>
<th>Time (months)</th>
<th>Amount of Dog Food (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>120</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td></td>
</tr>
</tbody>
</table>

c. Connect the points to show the shape of the graph. Do all of the points on the line make sense in this situation?
TALK the TALK

Let Us Organize the Ways

Multiple representations—including drawings or diagrams, verbal descriptions, tables, graphs, and equations—can be useful in analyzing and solving problems.

1. Complete the graphic organizer by describing the advantages of each representation.

- verbal
- table
- graph
- equation
VERBAL

TABLE

GRAPH

EQUATION

MULTIPLE REPRESENTATIONS

M3-204 • TOPIC 3: Graphing Quantitative Relationships
Assignment

Practice

1. Lashawna works at the local candy shop. The bulk candy is sold by the pound. Customers place the candy they would like to buy in a plastic bucket, and then Lashawna weighs it to determine how much the customer owes. Before calculating the price, Lashawna must subtract the weight of the plastic bucket. The candy bucket weighs 0.72 pound.
   a. Complete the table.
   b. Write an equation that models the relationship between the quantities in this situation.
   c. Use the table to create a graph of the relationship.
   d. Explain whether all points on the line make sense.

2. Lashawna is packaging some bulk candy for a sale. The price is $3.98 per pound.
   a. Write an equation to model the relationship between the total cost and the weight of the candy.
   b. Complete the table.
   c. Use the table to create a graph of the relationship.
   d. Explain whether it makes sense to connect the points on your graph.

Remember

When a quantity can have values that are only counting numbers, it is called a discrete quantity. When a quantity can have any value, it is called a continuous quantity.

Write

Create an algebraic equation. Represent the equation using a word problem, a table, and a graph.
Stretch
A cryptarithm is a puzzle which replaces digits with letters. Your job is to use reasoning to determine what digits the letters stand for. When two letters are the same, they represent the same digit. When the letters are different, they represent different digits.

In this famous cryptarithm, the sum is correct. Can you solve it?

\[
\begin{array}{c}
\text{S} \\
\text{E} \\
\text{N} \\
\text{D}
\end{array}
\quad +
\begin{array}{c}
\text{M} \\
\text{O} \\
\text{R} \\
\text{E}
\end{array}
\quad =
\begin{array}{c}
\text{M} \\
\text{O} \\
\text{N} \\
\text{E} \\
\text{Y}
\end{array}
\]

Review
Use the graph to estimate each solution.
1. How long did it take Serena to travel 70 miles?
2. How long did it take to burn 100 calories?

Write each statement as an algebraic expression.
3. five less than twice a number
4. seven and one half more than a number

Solve each equation.
5. \(20 = 6x\)
6. \(15.5 + p = 44\)
LEARNING GOALS

• Use multiple representations to solve one-step real-world and mathematical problems.
• Analyze the relationship between independent and dependent quantities using graphs, tables, and equations.
• Summarize the relationship between distance, rate, and time.

WARM UP

1. Express 3 hours and 15 minutes as a decimal.
2. Express 3 hours and 15 minutes in terms of minutes.
3. Express 2.75 hours in terms of hours and minutes.
4. Express 2.75 hours in terms of minutes.

You have graphed and analyzed a variety of relationships between two quantities. Some quantities are often grouped together. One set of such quantities is distance, rate, and time. What relationship exists between these quantities?
Gearing Up for the Olympics

Deazia has her sights set on competing in the triathlon in the 2024 Summer Olympics. A triathlon includes three sports: swimming, cycling, and running. Deazia must build up her endurance to be able complete all three events in quick succession!

As part of her training, Deazia will participate in a variety of triathlons over the next year. The table provides the distances for each leg of five different triathlons.

<table>
<thead>
<tr>
<th></th>
<th>Island Escape</th>
<th>Kid Zone</th>
<th>Olympic Style</th>
<th>Sprint</th>
<th>SuperTri</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swim</td>
<td>1.5 mi</td>
<td>600 m</td>
<td>1.5 km</td>
<td>750 m</td>
<td>2.4 mi</td>
</tr>
<tr>
<td>Cycle</td>
<td>18 mi</td>
<td>15 km</td>
<td>40 km</td>
<td>20 km</td>
<td>112 mi</td>
</tr>
<tr>
<td>Run</td>
<td>8 mi</td>
<td>5 km</td>
<td>10 km</td>
<td>5 km</td>
<td>26.2 mi</td>
</tr>
</tbody>
</table>

1. What is the total distance covered in each triathlon?

2. If Deazia completes all 5 triathlons in one year, how many miles will she swim during these competitions? How many miles will she cycle? How many miles will she run?
Swimming is the first leg of the triathlon, so Deazia has trained with a coach to improve her chance of getting off to a great start.

Deazia’s coach plotted her times and distances from her last few training sessions. Based on the data, the coach drew in a line to represent an approximation of her average speed.

1. Deazia’s little sister thinks that all of Deazia’s (time, distance) points should be on the line drawn by her coach. Is she correct? Explain your reasoning.
2. Use the graph to determine Deazia’s average swimming rate.

Deazia wants to know how long it will take her to complete the swimming segment of each triathlon. She decides to start with the Olympic Style triathlon, which has a 1.5 km swim leg.

**WORKED EXAMPLE**

\[
\frac{\text{distance}}{\text{time}} = \frac{1 \text{ km}}{20 \text{ minutes}}
\]

\[
\frac{1.5 \text{ km}}{\text{time}} = \frac{1 \text{ km}}{20 \text{ minutes}}
\]

\[
\times 1.5
\]

\[
\frac{1.5 \text{ km}}{\text{time}} = \frac{1 \text{ km}}{20 \text{ minutes}}
\]

\[
\times 1.5
\]

It should take Deazia 30 minutes to complete the swim segment of the Olympic Style triathlon.

3. Assuming Deazia swims at her average rate, determine how long it should take her to complete the swimming segment of the four remaining triathlons.
   
   a. Island Escape  
   b. Kid Zone  
   c. Sprint  
   d. SuperTri
4. Write an equation to represent the amount of time, \( t \), required for Deazia to swim a given distance, \( d \).

5. Write another equation to represent the distance, \( d \), Deazia can swim for a given amount of time, \( t \).

6. Deazia’s coach surprised her with an entry into a secret triathlon, the Mystic, last weekend.

   a. If she swam at her average rate and completed the swim segment in 45 minutes, how long was the swim segment? Explain your reasoning.

   b. How could you use a different strategy to verify your answer?
Deazia cycles as a regular part of her training schedule. After each ride, she records her distances and times in a table.

<table>
<thead>
<tr>
<th>Distance Biked (kilometers)</th>
<th>Time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1/4</td>
</tr>
<tr>
<td>35</td>
<td>3/4</td>
</tr>
<tr>
<td>90</td>
<td>1 1/2</td>
</tr>
</tbody>
</table>

1. Determine Deazia’s cycling rate in minutes per kilometer and in kilometers per minute.

2. Deazia would like to predict how long it will take her to complete the cycling segment of each triathlon. She thinks she should use the minutes per kilometer rate but her sister says that she should use the kilometers per minute rate. Who’s correct? Explain your reasoning.
3. Assuming she cycles at the same average rate, how long should it take Deazia to complete the cycling segment of each triathlon?
   a. Island Escape       b. Kid Zone
   c. Olympic Style       d. Sprint
   e. SuperTri

4. Write an equation to determine the amount of time required for Deazia to cycle a given distance.
5. Use your results to create a graph of the time, in minutes, that Deazia cycles versus the distance she cycles, in kilometers. Connect the plotted points.

6. You cannot graph the time and distance for the SuperTri on this graph. Explain how you know it would be on the line if the graph were extended.

7. In the Mystic triathlon, the cycling segment is 35 kilometers. Use your graph to estimate how long it should take Deazia to complete this segment of the triathlon. Explain your strategy.
Deazia runs every day as part of her training routine. She averages 9 minutes per mile.

1. Write an equation to determine the amount of time required to run a given distance.

2. Use your equation to determine how long will it take Deazia to complete the running segment of each triathlon.
   a. Island Escape  
   b. Kid Zone
   c. Olympic Style  
   d. Sprint
   e. SuperTri
3. Use your equation and the results from Question 2 to create a graph of time, in minutes, that Deazia runs versus her distance, in miles. Connect the plotted points.

4. After competing in the Mystic triathlon, Deazia reports that it took her 87 minutes to complete the running segment. Use the graph to estimate the length of the running segment of this triathlon.

5. Rewrite the equation in Question 1 in order to determine the distance traveled for a given amount of time.

Assume that Deazia runs at her average rate in this triathlon.
6. Use your new equation to determine the actual length of the running segment of the Mystic.

TALK the TALK

Reflecting on Triathlon Training

To analyze the three segments of the triathlons, you used distances traveled, times traveled, and rates.

1. Record the two equations that could be used to describe each leg of the race.
   - determine the distance, given the time
   - determine the time, given the distance

<table>
<thead>
<tr>
<th></th>
<th>Determine Distance</th>
<th>Determine Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swim</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cycle</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Run</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. What do you notice about the coefficients in the equations? Why does this make sense?
3. Write the ratio, including units, represented in the *Determine Distance* equation for each segment of the race.

4. Explain why the ratios, or rates, listed in Question 3, not their reciprocals, are the appropriate rates to use in determining distance.
Assignment

Write
Suppose your work partner was absent today. Write at least three sentences that summarize the relationship between distance \( d \), rate \( r \), and time \( t \). Be sure to talk about some of the multiple representations (verbal statements, graphs, tables, equations) of the relationship.

Remember
The equation that relates distance, rate, and time is often written as \( d = rt \).

Practice
1. An airplane takes off and climbs at a constant rate of 1400 feet per minute.
   a. Write an equation to model the relationship between the plane's altitude and the time in minutes.
   b. Complete the table.

<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Altitude (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td></td>
</tr>
</tbody>
</table>
   
   c. Use the equation to determine how much time it takes for the plane to reach an altitude of 3 miles.

2. A helium balloon rises at a constant rate of 200 feet per minute.
   a. Write an equation to model the relationship between the balloon's altitude and the time in minutes.
   b. Graph the equation.
   c. Use your graph to determine how much time it takes for the balloon to reach an altitude of 700 feet.

3. A car travels on the interstate at a constant speed. The distances are recorded in a table.

<table>
<thead>
<tr>
<th>Distance (miles)</th>
<th>Time (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.25</td>
<td>0.25</td>
</tr>
<tr>
<td>32.5</td>
<td>0.5</td>
</tr>
<tr>
<td>260</td>
<td>4</td>
</tr>
<tr>
<td>390</td>
<td>6</td>
</tr>
</tbody>
</table>
   
   a. Determine the car's rate in miles per hour and in hours per mile.
   b. Write an equation to determine the amount of time required to travel a given distance.
   c. Use the table to create a graph of the time versus the distance traveled.
   d. Determine how many minutes it will take the car to travel 43 miles.
Review
1. A business subtracts $7.50 from each employees’ gross weekly pay to cover the cost of their uniforms.
   a. Define variables for an employee’s gross weekly pay and for an employee’s weekly pay after the uniform fee.
   b. Write an equation that models the relationship between the variables.
   c. Graph the equation. Is the graph discrete or continuous?
   d. Calculate the gross weekly pay if the pay after the uniform fee was $67.23.

2. Determine each answer using the given formula.
   a. The formula $P = 4s$ is used to calculate the perimeter, $P$, of a square with a side length, $s$. Calculate the length of a side of the square if its perimeter is 34.56 inches.
   b. The formula $P = a + b + c$ is used to calculate the perimeter, $P$, of a triangle with side lengths $a$, $b$, and $c$. Calculate the unknown side length for a triangle with a perimeter of 52.81 inches and two sides measuring 16.32 inches each.

3. Calculate the area of each triangle.
   a.  
   b.  

Stretch
Alison and her friend are traveling home from New Jersey on Route 28. Alison thinks that taking Route 66 to Route 80 is a faster way home. Alison's friend says that staying on Route 28 is shorter, so they will make it home faster. Who’s correct? Which path is faster? By how much?
Graphing Quantitative Relationships Summary

KEY TERMS
- discrete graph
- continuous graph
- dependent quantity
- independent quantity
- independent variable
- dependent variable

LESSON 1
Every Graph Tells a Story

A **discrete graph** is a graph of isolated points. Often, those points are counting numbers.

A **continuous graph** is a graph with no breaks in it. Each point on a continuous graph, even those represented by fractional numbers, represents a solution to the graphed scenario.
When one quantity depends on another in a real-world problem situation, it is said to be the **dependent quantity**. The quantity on which it depends is called the **independent quantity**. The variable that represents the independent quantity is called the **independent variable**, and the variable that represents the dependent quantity is called the **dependent variable**.

For example, suppose you are draining a 150-gallon fish tank at a rate of 15 gallons per minute. How much water remains in the tank at a specific time?

In this scenario, the independent quantity is time, measured in minutes, and the dependent quantity is the number of gallons of water in the fish tank. The equation that represents the scenario is $w = 125 - 15t$. The independent variable is $t$, which represents the number of minutes, and the dependent variable is $w$, which represents the gallons of water in the tank.

Note that the independent quantity is plotted on the horizontal axis and the dependent quantity is plotted on the vertical axis.

![Graph showing water remaining in fish tank]

**The Power of the Horizontal Line**

You can use a graph to determine an independent quantity given a dependent quantity.

For example, Nic sells pretzels for $1.25 each morning at the games held at the Community Center. The amount of money collected for the number of pretzels sold can be represented by points on the graph. The equation corresponding to the graph is $y = 1.25x$. You can use the graph to determine how many pretzels Nic sold if he collected $10.

First, locate 10 on the $y$-axis and draw a horizontal line. This shows that $10$ is the amount of money collected. The $x$-value of the point where your horizontal line intersects with the graph of $1.25x$ is the number of pretzels sold for $10.
If you are given a graph, a solution to the equation represented by the graphed line is any point on that line. Nic sold 8 pretzels if he collected $10.

In some problem situations, when you model a relationship with a line, not all the points on the line will make sense. It is up to you to interpret the meaning of data values from the line drawn on a graph for each situation.

You can write an equation from a relationship given in a table.

For example, the number of tiles required to complete a job and the number of tiles ordered are represented in the table shown.

<table>
<thead>
<tr>
<th>Number of Tiles Required</th>
<th>60</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Tiles Ordered</td>
<td>80</td>
<td>95</td>
<td>120</td>
</tr>
</tbody>
</table>

The independent quantity is the number of tiles required to complete a job and the dependent quantity is the number of tiles ordered. By analyzing the table, you can see that the number of tiles ordered is always 20 more than the number of tiles required. An equation that models this relationship is $y = x + 20$. 
You can write an equation from a relationship represented in a graph.

For example, the graph shows the relationship between the distance of a train from the station and the time in minutes. A table of values can be completed using the points from the graph.

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>Distance (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>7</td>
<td>15</td>
</tr>
</tbody>
</table>

If \( t \) represents the time in minutes and \( d \) represents the distance from the station in miles, then the equation \( d = t + 8 \) represents the relationship between the quantities.

You can write an equation from a scenario.

For example, Deanna got a job working at the post office making $10.25 per hour.

An equation that models the relationship between the number of hours Deanna worked and the amount of money she earned can be written. Let \( a \) represent the amount Deanna earned and \( h \) represent the number of hours she worked. The equation is \( a = 10.25h \).
The equation that relates distance, rate, and time is often written as \( d = rt \).

For example, Deazia is training for a triathlon. Deazia’s coach plotted her times and distances from her last few swimming training sessions. Based on the data, the coach drew in a line to represent an approximation of her average speed.

Deazia’s swimming speed is a unit rate. There is a proportional relationship between distance and time.

Deazia wants to know how long it will take her to swim 1.5 km.

\[
\frac{\text{distance}}{\text{time}} = \frac{1 \text{ km}}{20 \text{ minutes}}
\]

\[
\frac{1.5 \text{ km}}{\text{time}} = \frac{1 \text{ km}}{20 \text{ minutes}} 
\times 1.5
\]

It should take Deazia 30 minutes to complete the swimming segment of the Olympic Style triathlon.
The lessons in this module extend your understanding of numbers and the number line to include negative numbers. You will use a number line to represent, make sense of, and order negative numbers. You will build on your knowledge of the coordinate plane to construct a four-quadrant graph. Throughout the module, you will analyze and solve a variety of real-world problems.

**Topic 1  Signed Numbers ...................................................... M4-3**

**Topic 2  The Four Quadrants .................................................. M4-53**
If you think of the surface of the ocean as 0, then a diver is in the negative numbers until he comes back up.

Lesson 1
Human Number Line
Introduction to Negative Numbers ........................................... M4-7

Lesson 2
Magnificent Magnitude
Absolute Value ................................................................. M4-23

Lesson 3
What's in a Name?
Rational Number System ................................................. M4-35
Carnegie Learning Family Guide
Course 1
Module 4: Moving Beyond Positive Quantities

TOPIC 1: SIGNED NUMBERS
In this topic, students are formally introduced to negative numbers. Students begin by reflecting the positive numbers across zero to build the rational number line. They focus on the meaning assigned to positive and negative rational numbers, with particular focus on the meaning of 0 in real-world and mathematical situations. Students develop an understanding of the relationship between opposites and distance on a number line, leading to the concept of absolute value. Throughout this topic, students continue to develop their fluency with whole numbers, fractions, and decimals.

Where have we been?
Prior to grade 6, students positioned whole numbers, fractions, and decimals on number lines and operated with these numbers using number lines as references. The opening activities in this topic draw on this prior knowledge of number lines and numbers’ positions relative to each other. In previous lessons in this course, students learned about and ordered non-negative rational numbers.

Where are we going?
Students will operate on signed numbers beginning in grade 7. The foundation provided in this topic will enable students to develop strategies for operating with signed numbers. Students will continue using the ideas from this topic throughout the remainder of the course. Just as they reflected the number line to include negative values, in the next topic students will reflect the first quadrant of a coordinate plane to create the four-quadrant coordinate plane.

Using a Number Line to Visualize Opposites
Each positive integer has an opposite, negative integer, and vice versa. The negative sign reflects a number across 0 on the number line. For example, the opposite of 3 is −3. Furthermore, the opposite of an opposite is the original number, e.g., −(−3) = 3.
Myth: Cramming for a test is just as good as spaced practice for long-term retention.

Everyone has been there. You have a big test tomorrow, but you’ve been so busy that you haven’t had time to study. So you had to learn it all in one night. You may have received a decent grade on the test. However, did you remember the material a week, month, or a year later?

The honest answer is, “probably not.” That’s because long-term memory is designed to retain useful information. How does your brain know if a memory is “useful” or not? One way is the frequency in which you encounter a piece of information. If you see something only once (like during cramming), then your brain doesn’t deem those memories as important. However, if you sporadically come across the same information over time, then it’s probably important. To optimize retention, encourage your student to periodically study the same information over expanding intervals of time.

#mathmythbusted

Talking Points
You can further support your student’s learning by resisting the urge, as long as possible, to get to the answer in a problem that your student is working on. Students are encountering negative numbers formally for the first time in this topic. They will need time and space to struggle with all the implications of working with this expanded number system. Practice asking good questions when your student is stuck.

Questions to Ask
• Let’s think about this. What are all the things you know?
• What do you need to find out?
• How can you model this problem?

Key Terms

**opposites**
Opposite numbers are reflections of each other across 0 on the number line.

**negative numbers**
The values to the left of zero on the number line are called negative numbers and are labeled with a negative sign.

**absolute value**
The absolute value of a number is its distance from zero on a number line.
You have used numbers equal to or greater than 0 to represent real-world situations. But how can you use numbers less than 0 to describe real-world situations?
Number Line Geography

1. What do you know about a number line?

2. Label the number line and be sure to include 0. Then plot and label a single point of your choice on the number line.
   
a. Draw a ray, or an arrow, beginning at your point to represent the numbers larger than the value at your point.

   b. Draw a ray, or an arrow, beginning at your point to represent the numbers smaller than the value at your point.

   c. At the ends of a number line, there are arrows going in both directions. What do these arrows indicate?

   d. What do you think is on the number line to the left of 0?
Let's use a number line to represent time.

Your teacher will assign students to participate in the activity. Be sure to record what happens on the number line.

1. For each student, plot and label the point where the student stands on the number line. Also identify what time is represented by the point.

**Student A:** Stand at 0 to represent the time right now.
**Student B:** Stand at the point that represents 3 hours from now.
**Student C:** Stand at the point that represents 3 hours ago.
**Student D:** Stand at the point that represents 5 hours from now.
**Student E:** Stand at the point that represents 2 hours ago.
**Student F:** Stand at the point that represents 7 hours ago.

A number line can be created by reflecting the positive numbers across zero. The values to the left of zero on the number line are called **negative numbers** and are labeled with a negative sign. The positive values extend to positive infinity, and the negative numbers extend to negative infinity. **Infinity**, represented by the symbol $\infty$, means a quantity with no end or bound. The number line goes on forever in both directions!

A negative number is written with a negative sign. You can write a positive number with a positive sign or without any sign. For example, positive 5 can be written as $+5$ or $5$. 

![Diagram of a number line with infinity symbols at both ends, showing points representing different times for each student.](image-url)
2. Describe the change in the values of the numbers as you move to the right on the number line.

3. Describe the change in the values of the numbers as you move to the left on the number line.

Consider your class time number line.

4. Describe the locations of the points that represent time in the future.

5. Describe the locations of the points that represent time in the past.

6. How would your number line be labeled differently from one created by a class that starts at a different time?

7. What observations can you make about where a given number of hours before or after time 0 is plotted? What do you notice about its distance from 0? For example, what do you notice about 3 hours before and 3 hours after now? Or 6 hours before and 6 hours after now?
Let’s think more about both sides of 0 on a number line.

Your teacher will model a number line.

1. Create and label a number line according to the model.

2. Plot and label the location where each student stands on the number line. In the table, identify the value represented by the location where the student is standing.

   Student A: Stand at 0.
   Student B: Stand at 4.5.
   Student C: Stand at the opposite of 4.5.
   Student D: Stand at −6.
   Student E: Stand at the opposite of −6.
   Student F: Stand at a location between 2 and 3.
   Student G: Stand at the location that is the opposite of Student F.

3. Describe the number line relationship of the students who were opposites of each other.
Opposite numbers are reflections of each other across 0 on the number line.

- The opposite of a positive number is a corresponding negative number.
- The opposite of a negative number is a corresponding positive number.

Attaching a negative sign to a number means reflecting that number across 0 on the number line.

4. Use symbols to represent the opposite of 4.5 and the value it represents.

\[-(4.5) = \____\]

5. Use symbols to represent the opposite of \(-6\) and the value it represents.

\[-(-6) = \____\]

6. What do you notice about the distance from 0 of corresponding opposite numbers?

7. What is the opposite of 0?

8. Name the opposite of each number. Then, plot each number and its opposite on the number line.

a. \(1\frac{1}{2}\)  
b. \(-5\)  
c. \(-9.9\)
Alyson and her friends are trying to decide if they can go to the movies. Each ticket costs $9.00. After checking their wallets, each friend comments on how much money they have.

- Alyson: I have $2.50 more than the movie costs.
- Sharon: Oh, I don’t have enough money. I’m $4.00 short.
- Brian: Not only can I buy a ticket, but I have just enough money to buy the $8.00 snack combo!
- Eileen: If I can find one more quarter, I can go.

Myron and Paulie created different number lines to represent the scenario.

1. What does each point represent on Myron’s number line?

2. What does each point represent on Paulie’s number line?
3. Myron and Paulie are thinking about 0 differently. Explain what 0 represents on each number line.

4. Suppose the four friends decide to go to a matinee instead, where the ticket price is $7.50.
   a. How would Myron’s number line change?
   b. How would Paulie’s number line change?
Number lines can also be vertical, like a thermometer or a measure of elevation.

1. Discuss and write a sentence to describe the meaning of each statement.

   a. The weather forecaster predicts the temperature will be below zero.

   b. A submarine travels at 3000 feet below sea level.

   c. Badwater Basin in Death Valley, California, is 86 meters below sea level.

2. Mark each temperature on the thermometer shown.

   a. The highest temperature on record in the United States is 134°F. It occurred in 1913 in Death Valley, California.

   b. The lowest temperature on record is −80°F. It occurred at Prospect Creek Camp, Alaska.

   c. The lowest temperature recorded in the contiguous 48 states is −70°F. It occurred in Montana.

   d. The highest winter average temperature in the United States is 78°F, which occurs in Honolulu, Hawaii.
3. Which is colder, the lowest temperature recorded in Alaska or the lowest temperature recorded in Montana? How do you know?

4. Yadi and Eric were comparing 25 degrees to −27 degrees.

- Yadi wrote $25 < -27$ and justified her comparison by stating that the further a number is from zero, the greater the number.

- Eric wrote $25 > -27$ and justified his comparison by stating that the greater temperature will be above the second temperature on a thermometer.

Who is correct? Explain your choice.

5. Plot each set of temperatures on the thermometer. Then insert a $>$ or $<$ symbol to make each number sentence true.

   a. $-26^\circ F \underline{\hspace{1cm}} -31^\circ F$

   b. $-6^\circ F \underline{\hspace{1cm}} -17^\circ F$

   c. $-9^\circ F \underline{\hspace{1cm}} 8^\circ F$

6. Order the temperatures from least to greatest.

   $25^\circ F \quad -33^\circ F \quad 0^\circ F \quad 105^\circ F \quad -40^\circ F \quad -5^\circ F \quad 67^\circ F$
Helen and Grace started a company called Top Notch. They check the company’s bank balance at the end of each week. The table shown represents the first 10 weeks of operation. Overdrafts, or weeks when they owe the bank money, are represented by amounts within parentheses. For example, ($25) denotes an overdraft of $25; they owe the bank $25. Amounts that are not in parentheses are when they made money.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance</td>
<td>$159</td>
<td>($201)</td>
<td>$231</td>
<td>($456)</td>
<td>($156)</td>
<td>($12)</td>
<td>$281</td>
<td>$175</td>
<td>$192</td>
<td>$213</td>
</tr>
<tr>
<td>+/- Number</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Use the table and number line to answer each question.

   a. Write each as a positive or negative number and then plot the number on the number line.

   b. What does 0 represent in this situation?

   c. In which week did they have the highest bank balance?

   d. In which week did they show greatest overdraft?

2. For each pair of weeks, write an inequality statement to compare the positive and negative numbers. Interpret the statement in context.

   a. Week 1 and Week 5

   b. Week 4 and Week 6
You can compare different types of numbers by plotting the numbers on a number line.

3. Use the number line to answer each question.

   a. Plot each value on the number line.

   \[
   -6\frac{2}{3}, -20, 0, 10.5, -17\frac{1}{2}, -7.98, 12, -3, -13
   \]

   b. Which of the numbers has the least value? How do you know?

   c. Which of the numbers has the greatest value? How do you know?

   d. Order the numbers from least to greatest.

4. Plot each rational number on the number line. Then, insert a >, <, or = symbol to make each number sentence true.

   a. \(-10.25 \quad \quad \quad \quad \quad -15\frac{2}{3}\)

   b. \(-17 \quad \quad \quad \quad \quad -17\)

   c. \(5\frac{2}{3} \quad \quad \quad \quad \quad -8.28\)
TALK the TALK

Putting It All Together

1. What does 0 mean on a number line?

2. What does opposite mean in terms of a number line?

3. Compare the types of numbers. Use what you know about number lines to explain your reasoning.
   a. Which is greater—a negative or a positive rational number?
   b. Which is greater—zero or any positive rational number?
   c. Which is greater—zero or any negative rational number?
   d. How do you decide which of two numbers is greater if both numbers are positive?
   e. How do you decide which of two numbers is greater if both numbers are negative?
4. Your sixth grade cousin goes to school in a different state. His math class has not yet started comparing integers. Write him an email explaining how to compare any two numbers. Be sure to include 1 or 2 examples and enough details that he will be able to explain it to his class.
**Assignment**

**Write**
Write a sentence to explain the relationship between opposites and negative numbers.

**Remember**
The rational number line is used to represent positive numbers, negative numbers, and zero. The values to the left of zero on the number line are reflections of the values on the right across 0.

**Practice**

1. Plot each number and its opposite on the number line.
   - a. \(-1\)
   - c. \(\frac{3}{4}\)
   - e. 0.009
   - b. 0.1
   - d. \(-1.9\)

2. Order the numbers from least to greatest.
   
   \[
   0.125 \quad 0.1 \quad -\frac{4}{9} \quad \frac{4}{11} \quad -\frac{3}{2} \quad -2.75
   \]

3. The Ravine Flyer II is a steel and wood roller coaster that takes advantage of the terrain in Erie, PA, to make the ride more exciting. Although the coaster is only 80 feet high, it follows the line of a cliff in order to drop to \(-35\) feet (0 represents the height of the cliff).
   - a. Plot the highest and lowest points of the roller coaster on a vertical number line.
   - b. Explain why a vertical number line better represents the problem context than a horizontal number line.
   - c. How many total feet does the roller coaster drop?

4. The Monster is a roller coaster that uses a design similar to the Ravine Flyer II. The Monster reaches a height of 120 feet, but then drops to \(-25\) feet.
   Use number line to order the highest and lowest points of the two roller coasters from least to greatest.

5. An amusement park wants to design a coaster that rises 60 feet above ground and then drops the same distance below ground through a tunnel. Represent the underground depth with a number, and explain its relationship with the above ground height.
Stretch
Create a new situation, similar to Activity 1.3 *Representing Money on a Number Line*, in which zero can have two different meanings.

Review
Name the two quantities that are changing in each and determine which quantity is the dependent quantity and which is the independent quantity.
1. Terrence types 80 words per minute.
2. To determine the total weekly wages of his employees, Mr. Jackson multiplies the total number of hours his employees work by $12.
3. A mountain climber is ascending a mountain at a rate of 5 feet per minute. Define variables and write an equation that represents the situation. Graph the equation on a coordinate plane.

Perform the indicated operation.
4. $11\frac{4}{5} + 5\frac{2}{3}$
5. $\frac{27}{4} \div \frac{3}{2}$
WARM UP
Plot each set of numbers on the number line and describe the relationship between the numbers.
1. 5 and −5
2. $2\frac{3}{4}$ and $-2\frac{3}{4}$
3. 8.634 and −8.634

LEARNING GOALS
• Explain the meaning of the absolute value of a rational number as its distance from 0 on a number line.
• Interpret the meaning of absolute value as the magnitude for a positive or negative quantity in a real-world context.
• Evaluate the absolute value of a quantity.
• Compare and order numbers expressed as absolute value and distinguish absolute value comparisons from statements about order.

KEY TERM
• absolute value

Numbers can be described by their distance from 0 on the number line. How can you use these distances to solve real-world problems?

LESSON 2: Magnificent Magnitude • M4-23
1. Plot a point at \(-7\) on the number line.

2. Describe the distance from \(-7\) to 0.

3. Plot as many other points as possible on the number line that are the same distance from 0 as \(-7\).

4. How many numbers did you plot? Why do you think this is true?
Let’s revisit the number line from the *Human Number Line* lesson.

Your teacher will assign students to participate in the activity. Be sure to record what happens on the number line.

- Student A: Stand on 0 and hold one end of the string provided by your teacher.
- Student B: Hold the other end of the string and stand on the number line as far as possible from Student A. Are there other places on the number line that you could stand and be as far from Student A as possible?
- Repeat this activity with two more pieces of string of different lengths and two additional students, Students C and D. Student A will hold the 0 end of each string.

1. Compare the locations where each student stood.
   a. What do you notice about the distances each time the students moved?
   b. What do you notice about the approximate values for the numbers where each stood?

The magnitude, or absolute value, of a number is its distance from zero on a number line. The symbol for absolute value is \(|n|\). The expression \(|n|\) is read as “the absolute value of a number n.”

Because distance cannot be negative, the absolute value of a number is always positive or 0.
2. Plot 5 on the number line.
   a. How far is 5 from 0?  
   b. \(|5| = __\)

3. Plot \(-7.2\) on the number line.
   a. How far is \(-7.2\) from 0?  
   b. \(|-7.2| = __\)

4. Explain what each statement means. Name any other values that have the same absolute value, if possible.
   a. \(|-5|\)  
   b. \(|1 \frac{5}{6}|\)
   c. \(|0.75|\)  
   d. \(|-1.36|\)

Use your investigation and a number line to answer each question.

5. Can two different numbers have the same absolute value? If so, provide examples.

6. What can you say about the absolute value of
   a. any positive number?
   b. any negative number?
   c. zero?
Absolute values are used in real-world applications when you are interested in only the number and not in the sign of the number. When you look at temperature changes, you could say the temperature “fell by,” “decreased by,” or “increased by” an absolute value.

1. Complete the table with an appropriate situation, absolute value statement, and/or number. For the last row, assign the correct units to the number based on your situation.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Absolute Value Statement</th>
<th>Numeric Example (with units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The temperature went from 55°F to 5°F</td>
<td>The temperature fell by 50°F.</td>
<td>−50°F</td>
</tr>
<tr>
<td>The bank account balance went from $2500 to $2250.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The bank account balance went from $495 to $615.</td>
<td></td>
<td>$120</td>
</tr>
<tr>
<td>During the hike, the elevation went from 1125 feet to 1750 feet.</td>
<td>The water level increased by 4.9 feet.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>−10 feet</td>
</tr>
</tbody>
</table>
You also use absolute value statements to describe how numbers compare with other numbers. You often use these statements without thinking about “less than” or “greater than.” Rather, you use words like “debt,” “lost,” “colder,” “depth,” “above,” “hotter,” or “below.”

2. Complete the table with an appropriate situation, absolute value statement, and/or example. For the last row, assign the correct units to the numeric example based on your situation.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Absolute Value Statement</th>
<th>Numeric Example (with units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A water level less than $-2\frac{1}{2}$ feet</td>
<td>More than $2\frac{1}{2}$ feet below a full pool</td>
<td>$-3$ feet</td>
</tr>
<tr>
<td>An account balance less than $-30$</td>
<td>A debt greater than $30$</td>
<td></td>
</tr>
<tr>
<td>A weight less than $-7.5$ pounds of previous weight</td>
<td>Lost more than 7.5 pounds</td>
<td></td>
</tr>
<tr>
<td>A dive to a height less than $-350$ feet</td>
<td>Colder than 10 degrees below 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A depth greater than 15 m</td>
<td></td>
</tr>
<tr>
<td></td>
<td>A golf tournament stroke total more than 7 strokes below par</td>
<td>$-100$ stroke below par</td>
</tr>
</tbody>
</table>

Par is the number of strokes, or swings, a golfer is expected to take.
ACTIVITY 2.3

Using Absolute Value to Solve Real-World Problems

1. In many buildings, particularly outside of the United States, the ground floor of a building is labeled as G or Lobby. The first floor of the building is one floor above the ground floor. The building pictured has a lobby, 10 floors of offices, and 4 floors of garage below the lobby.

   a. Melanie has an office on the 9th floor and parks on the 3rd floor below the ground floor. Taylor and Cecelia are determining how many floors Melanie must go up from her car to reach her office.

   Taylor represents the 9th floor as 9 and the 3rd floor below ground as \(-3\). Therefore, since \(9 - 3 = 6\), Melanie traveled 6 floors to get from her car to her office.

   Cecelia says that the ground floor to the 9th floor is 9 floors, and from the ground floor to the 3rd garage level is 3 floors. Melanie traveled \(|9| + |-3| = 9 + 3 = 12\) floors.

   Who is correct? Explain your reasoning.
Write a numeric expression using absolute values that would represent each situation. Then calculate the answer.

b. Caleb parks his car on the 2nd floor below ground and works on the 7th floor. How many floors must he go up from his car to reach his office?

c. Lucinda is working on the 8th floor. At lunch, she goes to her car on the 4th floor below ground, and then back up to the lobby. How many total floors does Lucinda travel?

d. If Damon goes from his office on the 10th floor to a meeting on the 5th floor, how many floors does he travel and in which direction?

2. The Top Notch company’s bank balances are shown. The table represents the first 10 weeks of operation. Overdrafts are represented by amounts within parentheses.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance</td>
<td>$159.25</td>
<td>($201.35)</td>
<td>$231.57</td>
<td>($456.45)</td>
<td>($156)</td>
<td>($12.05)</td>
<td>$281.34</td>
<td>$175</td>
<td>$192.34</td>
<td>$213</td>
</tr>
</tbody>
</table>

a. Use estimation to determine the gains/losses between consecutive weeks.

b. Between which two weeks did Top Notch have the largest gain in money? What was the actual gain?

c. Between which two weeks did Top Notch have the largest loss in money? What was the actual loss?

d. What was the difference between the company’s lowest balance and its highest balance?
e. Order the estimated gains and losses that you determined in part (a) from least to greatest. Use a negative sign to indicate losses.

f. Order the estimated gains and losses that you determined in part (a) from least to greatest according to their absolute values. What does the absolute value mean in the context of this problem?

g. Why are the orders different in parts (e) and (f)?

3. As part of a long-term science experiment, two rulers were connected at zero and used to measure the water level in a pond. The connected rulers were placed in the pond so that the water level aligned at zero. The water level was measured each week for 10 weeks.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
</table>

a. What do the positive numbers represent? What do the negative numbers represent?

b. Between which two weeks did the water level change the most? What was the change?

c. Between which two weeks did the water level change the least? What was the change?

d. How much did the water level change between Weeks 4 and 5? What was the change?
You Absolutely MUST Compare These!

Insert a $>$, $<$, or $=$ symbol to make each statement true. Justify each answer in terms of the definition of absolute value and number lines.

1. $|-4.67| \text{ ___ } |3|$

2. $|-15| \text{ ___ } |15|$

3. $|25 \frac{5}{10}| \text{ ___ } |33 \frac{2}{3}|$

4. $|13.45| \text{ ___ } |27|$

5. $|-15.34| \text{ ___ } |-11 \frac{1}{12}|$

6. $|-19 \frac{1}{2}| \text{ ___ } |5.5|
Practice
1. Julio is a wrestler for his high school wrestling team in the winter. Julio needs to stay around 140 pounds in the off-season. He charted his weight over the summer by listing the differences his weight was from 140 pounds. He uses negative numbers when his weight was under 140 pounds and positive numbers when his weight was above 140 pounds.

<table>
<thead>
<tr>
<th>Week</th>
<th>Weight Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+4.5</td>
</tr>
<tr>
<td>2</td>
<td>+2.1</td>
</tr>
<tr>
<td>3</td>
<td>−1.5</td>
</tr>
<tr>
<td>4</td>
<td>−0.5</td>
</tr>
<tr>
<td>5</td>
<td>−2.5</td>
</tr>
<tr>
<td>6</td>
<td>+1.5</td>
</tr>
<tr>
<td>7</td>
<td>−3.75</td>
</tr>
<tr>
<td>8</td>
<td>−2.8</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>+1.3</td>
</tr>
<tr>
<td>11</td>
<td>−1.5</td>
</tr>
<tr>
<td>12</td>
<td>−5</td>
</tr>
</tbody>
</table>

a. Was the amount his weight varied from 140 pounds in week 4 more or less than the amount it varied from 140 pounds in week 8?
   Insert a $>$, $<$, or $=$ symbol to make the statement true. Explain your answer.
   $|−0.5| \quad \bigcirc \quad |−2.8|$

b. Was the amount his weight varied from 140 pounds in week 6 more or less than the amount it varied from 140 pounds in week 11?
   Insert a $>$, $<$, or $=$ symbol to make the statement true. Explain your answer.
   $|+1.5| \quad \bigcirc \quad |−1.5|$

c. Use absolute values to determine the difference in Julio’s weight from week 7 to week 10.
d. Use absolute values to determine the difference in Julio’s weight from week 8 to week 12.

2. The table shown tracks Julio’s weight changes that he reports to his coach for the first 4 weeks of school. Complete the table to explain the changes.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Absolute Value Statement</th>
<th>Rational Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>His weight went from 140 to 135 pounds.</td>
<td>His weight fell by 5 pounds.</td>
<td></td>
</tr>
<tr>
<td>His weight went from 135 pounds to 141 pounds.</td>
<td></td>
<td>6 lb</td>
</tr>
<tr>
<td>His weight went from 141 pounds to 140.5 pounds.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>His weight went from 140.5 pounds to 139 pounds.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

LESSON 2: Magnificent Magnitude   •   M4-33
3. Weather experts collect many types of data to study and analyze, including extreme temperature changes. The interior West of North America experiences great temperature changes due to Chinook Winds. The table shows extreme temperature rises in three cities.

<table>
<thead>
<tr>
<th>Place</th>
<th>Granville, ND</th>
<th>Fort Assiniboine, MT</th>
<th>Spearfish, SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Date</td>
<td>Feb. 21, 1918</td>
<td>Jan. 19, 1892</td>
<td>Jan. 22, 1943</td>
</tr>
<tr>
<td>Time Period</td>
<td>12 hours</td>
<td>15 minutes</td>
<td>2 minutes</td>
</tr>
<tr>
<td>Temperature</td>
<td>From −33°F to 50°F</td>
<td>From −5°F to 37°F</td>
<td>From −4°F to 45°F</td>
</tr>
</tbody>
</table>

For each city, write an absolute value equation and use it to determine how much the temperature rose.

a. Granville, ND  
b. Fort Assiniboine, MT  
c. Spearfish, SD

4. Tyler measured the rainfall and evaporation using a rain gauge in his backyard for 8 days. Tyler marked his rain gauge with values from −6 inches to +6 inches and filled the gauge with water to the zero mark. For each question, write an expression using absolute value and then calculate the answer.

<table>
<thead>
<tr>
<th>Days</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauge Reading</td>
<td>0.5</td>
<td>−1.3</td>
<td>3.7</td>
<td>4.2</td>
<td>2.1</td>
<td>−0.9</td>
<td>−2.4</td>
<td>5.6</td>
</tr>
</tbody>
</table>

a. On how many days out of the eight did it rain?  
b. Between which two consecutive readings did it rain the most? How many inches were recorded?  
c. Between which two consecutive readings was evaporation the greatest? How many inches of water evaporated?  
d. Calculate the gain or loss of water in the rain gauge between days 1 and 2. Express the change in the water level in the gauge as a positive or negative number.  
e. Calculate the gain or loss of water in the rain gauge between days 2 and 3. Express the change in the water level in the gauge as a positive or negative number.

**Stretch**

Write a scenario to represent each rational number.  
1. −12  
2. −4 \( \frac{1}{2} \)  
3. 7.3  
4. −0.7

**Review**

1. Use the >, <, or = symbol to complete each statement.  
   a. −5 \( \square \) −8  
   b. −3 \( \square \) 0  
   c. 5 \( \square \) −5

2. Five employees work on the receiving dock at a factory. They divide the number of crates they unload from each truck equally. Define variables for the number of crates on a truck and for the number of crates each employee unloads from the truck. Write an equation that models the relationship between these variables.

3. Solve for the variable in each equation.  
   a. \( \frac{5}{2} = 15 \)  
   b. \( y - 8 = 19 \)
LESSON 3: What's in a Name?

Rational Number System

WARM UP
Represent each decimal or percent as a fraction in lowest terms.

a. 0.3
b. 2.8
c. \(\frac{3}{4}\%\)
d. 212%

LEARNING GOALS
• Classify numbers according to their number systems.
• Apply and extend an understanding of whole numbers and integers to the system of rational numbers.
• Understand ordering of rational numbers.

KEY TERMS
• integers
• ellipsis
• rational numbers
• Density Property

You use many different types of numbers in math class and in the world, including whole numbers, fractions, and decimals, both positive and negative. How can you organize and classify different types of numbers?
Sort It Out!

Cut out the cards found at the end of the lesson. Then, analyze and sort the numbers into different groups. You may group them in any way you feel is appropriate, but you must sort the numbers into more than one group.

1. For each of your groups,
   - create a title that fits the numbers in that group.
   - list the numbers included.
   - write a rationale for why you group those particular numbers.

2. Compare your sort with your classmates’ sorts. Create a list of the different ways your class grouped the numbers.
1. Suzanne grouped these numbers together. Why do you think she put these numbers in the same group?

   \[0, -452, 9, 24, |-3|, -3, -(−9), |-452|\]

2. Zane had a group similar to Suzanne’s but he did not include \(-452\) and \(-3\). Why do think Zane omitted these numbers from his group?

3. Amelia said that she created two groups: Group 1 contains all the numbers that can be written as fractions and Group 2 contains all the numbers that cannot be written as fractions. Analyze Amelia’s sorting idea.

   a. Which numbers do you think Amelia placed in Group 2?

   b. Justine is not sure about Amelia’s sort. She thinks that all of the numbers can be written as fractions. Is Justine correct? Explain why or why not.
You have used different sets of numbers, including the set of natural, or counting, numbers and the set of whole numbers.

4. Identify the numbers from the sort that are in each set.
   a. natural numbers
   b. whole numbers

Throughout this topic, you have been learning about the set of integers. **Integers** are the set of whole numbers with their opposites. The integers can be represented by the set \{..., −3, −2, −1, 0, 1, 2, 3, ...\}.

5. Identify the numbers from the sort that are included in the set of integers.

You have also worked with rational numbers throughout this year. **Rational numbers** are the set of numbers that can be written as \( \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \neq 0 \).

6. Identify the numbers from the sort that are included in the set of rational numbers.
There are many ways you can classify numbers. As you saw in the previous activity, many of the classifications are subsets of other classifications. The diagram shows the different sets of numbers you have encountered in your mathematical experiences.

Natural numbers are a subset of whole numbers.

Whole numbers are a subset of integers.

Integers are a subset of rational numbers.

Pin the number on the bullseye! Your teacher will direct students to pin (or tape) a number card to its correct location in the diagram of the rational number set.

1. For each value, check all of the number sets to which it belongs.

<table>
<thead>
<tr>
<th>Number</th>
<th>Natural Number</th>
<th>Whole Number</th>
<th>Integer</th>
<th>Rational Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.222</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−4.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{3}{5}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>54</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{23}{3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.667</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−1,364,698</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. Complete the table with the missing examples and descriptions.

<table>
<thead>
<tr>
<th>Examples</th>
<th>Natural Numbers</th>
<th>Whole Numbers</th>
<th>Integers</th>
<th>Rational Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
<td>Counting numbers</td>
<td>Natural numbers and 0</td>
<td>... , -3, -2, -1, 0, 1, 2, 3, ...</td>
<td></td>
</tr>
</tbody>
</table>

ACTIVITY 3.3

Density

The **Density Property** states that between any two rational numbers there is another rational number. The property is not true for natural numbers, whole numbers, or integers. For example, there is no integer between 25 and 26. There is no whole number or natural number between 12 and 13.

1. Plot the given rational numbers. Then plot and label a rational number between each pair of rational numbers.

   a. \( \frac{4}{3} \) and \( \frac{2}{3} \)

   ![Graph of \( \frac{4}{3} \) and \( \frac{2}{3} \)]

   b. 5.5 and 5.6

   ![Graph of 5.5 and 5.6]

   c. 0.45 and 0.46

   ![Graph of 0.45 and 0.46]

   d. \(-0.45\) and \(-0.46\)

   ![Graph of \(-0.45\) and \(-0.46\)]
Complete each rational number line with a partner.

2. Create a number line from 0 to 1. Your goal is to plot and label a rational number closer to 1 than your partner.

   Partner 1: Plot a rational number, \( A \), between 0 and 1 that is close to 1.

   Partner 2: Plot a rational number, \( B \), between \( A \) and 1.

   Repeat at least 2 more times.

3. Create a number line from \(-1\) to 0. Your goal is to plot and label a rational number closer to 0 than your partner.

   Partner 1: Plot a rational number, \( A \), between \(-1\) and 0 that is close to 0.

   Partner 2: Plot a rational number, \( B \), between \( A \) and 0.

   Repeat at least 2 more times.

4. Create a number line from \(-6\) to \(-5\). Your goal is to plot and label a rational number closer to \(-5\) than your partner.

   Partner 1: Plot a rational number, \( A \), between \(-6\) and \(-5\) that is close to \(-5\).

   Partner 2: Plot a rational number, \( B \), between \( A \) and \(-5\).

   Repeat at least 2 more times.
TALK the TALK

Do They Always Belong?

Determine if each statement is true or false. Justify your answer using definitions and/or examples.

1. True  False  All whole numbers are rational numbers.

2. True  False  All rational numbers are whole numbers.

3. True  False  All rational numbers are integers.

4. True  False  All integers are rational numbers.

5. True  False  All whole numbers are integers.

6. True  False  All integers are whole numbers.
<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>−5.78</td>
<td>$2\frac{15}{16}$</td>
<td>$\frac{3}{4}$</td>
<td>−452</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>24</td>
<td>9</td>
<td>$\frac{6}{7}$</td>
<td>$−\frac{6}{7}$</td>
</tr>
<tr>
<td>−0.5</td>
<td>0.5</td>
<td>$−\frac{1}{2}$</td>
<td>2.5%</td>
<td>5.78</td>
</tr>
<tr>
<td>−3</td>
<td>$</td>
<td>−3</td>
<td>$</td>
<td>$−\frac{2}{3}$</td>
</tr>
<tr>
<td>−6.41</td>
<td>$</td>
<td>6.41</td>
<td>$</td>
<td>$−(−9)$</td>
</tr>
<tr>
<td>225%</td>
<td>$6\frac{1}{4}$</td>
<td>25%</td>
<td>0.25%</td>
<td>$</td>
</tr>
</tbody>
</table>
Assignment

**Write**
Define each term in your own words.
1. The set of rational numbers
2. The Density Property

**Practice**
1. Write all the sets of numbers to which each value belongs.
   a. The tundra covers about $\frac{1}{5}$ of Earth’s surface.
   b. The average annual temperature is $-18^\circ$ Fahrenheit.
   c. There are 48 varieties of land mammals found in the tundra region.
   d. The permafrost is a layer of frozen soil that is located below Earth’s surface at $-1476$ feet.
   e. During the summer months, the low temperature averages about 37.4° F.

2. Nadine collects data about some animals.
   Determine a rational number between each pair of rational numbers. Plot all three numbers on a number line.
   a. A mole’s runway is between $-3$ and $-12$ inches in the ground.
   ![Number line with values -12 to 0]
   b. The musky rat kangaroo weighs between $\frac{3}{4}$ and $\frac{3}{2}$ pound.
   ![Number line with values 0 to 2]
   c. The percent of change of the Alaskan polar bear population in the past year was between $-0.33$ and $-0.32$.
   ![Number line with values -0.4 to -0.3]

**Remember**
Rational numbers include all numbers that can be written in the form $\frac{a}{b}$, where $a$ and $b$ are integers and $b$ is not zero.

**LESSON 3:** What’s in a Name? • M4-45
**Stretch**

Are there more integers or more natural numbers? Even though there are infinitely many of both, it seems like there should be more integers than natural numbers. But, actually, there are just as many integers as there are natural numbers!

If you can show how to assign an integer to every natural number, you will demonstrate that the two sets of numbers are equal. How do you think this can be done?

**Review**

1. Write an absolute value expression to calculate the answer to each question.
   a. The temperature at 9:00 A.M. was 40°F. The temperature at 2:00 P.M. was −10°F. What was the change in temperature?
   b. You began your hike at 30 feet below sea level. You are now at 200 feet. How far have you hiked?

2. Complete the table for the equation \( w = \frac{m}{9.2} \).

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>27.6</td>
<td></td>
</tr>
<tr>
<td>74.52</td>
<td>5</td>
</tr>
<tr>
<td>92</td>
<td>14</td>
</tr>
</tbody>
</table>

3. Plot each ordered pair on a coordinate plane.
   a. (2, 4)
   b. (5.5, 1.75)
   c. \( \left( \frac{4}{3}, \frac{5}{5} \right) \)
Signed Numbers Summary

KEY TERMS

- negative numbers
- infinity
- absolute value
- integers
- ellipsis
- rational numbers
- Density Property

LESSON 1

Human Number Line

A number line can be created by reflecting the positive numbers across zero. The values to the left of zero on the number line are called negative numbers and are labeled with a negative sign. You can write a positive number with a positive sign or without any sign. For example, positive 5 can be written as +5 or 5.

The positive values extend to positive infinity, and the negative numbers extend to negative infinity. Infinity, represented by the symbol ∞, means a quantity with no end or bound.
Opposite numbers are reflections of each other across 0 on the number line.

- The opposite of a positive number is a corresponding negative number.
- The opposite of a negative number is a corresponding positive number.

Attaching a negative sign to a number means reflecting that number across zero on the number line. The number 0 is the only number that doesn’t have an opposite.

For example, the numbers $\frac{6}{2}, -13, -18.5$, and their opposites are plotted on the number line.

Number lines can also be vertical, like a thermometer or a measure of elevation.

You can use a thermometer to plot temperatures and to compare and order temperatures. In vertical number lines like this one, the greater the value, the higher up on the number line.

For example, compare 40 degrees to $-60$ degrees. By plotting each temperature on the thermometer, you can see that 40 degrees is above $-60$ degrees. Therefore $40 > -60$.

You can compare different types of numbers by plotting the numbers on a number line.

For example, the numbers, $-\frac{62}{3}, 10.5, -25, 17$, and 0 have been plotted on the number line. Use the number line to order the values from least to greatest.

From the number line you can determine that $-25$ has the least value because it is the farthest to the left and 17 has the greatest value because it is farthest to the right. The numbers ordered from least to greatest are $-25, -\frac{62}{3}, 0, 10.5$, and 17.
The magnitude, or **absolute value**, of a number is its distance from zero on a number line. The symbol for absolute value is | |. The expression |n| is read as “the absolute value of a number n.” Because distance cannot be negative, the absolute value of a number is always positive or 0.

|9| = 9, because 9 is 9 units from 0 on a number line.

|−3.8| = 3.8, because −3.8 is 3.8 units from 0 on a number line.

Absolute values are used in real-world applications when you are interested in only the number and not in the sign of the number. You also use absolute value statements to describe how numbers compare with other numbers.

<table>
<thead>
<tr>
<th>Situation</th>
<th>Absolute Value Statement</th>
<th>Numeric Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>The temperature went from 55°F to 5°F.</td>
<td>The temperature fell by 50°F.</td>
<td>−50°F</td>
</tr>
<tr>
<td>The bank account balance went from $550 to $795.</td>
<td>The balance increased by $245.</td>
<td>$245</td>
</tr>
<tr>
<td>A water level went from 10.3 feet to 6.7 feet.</td>
<td>A water level fell by 3.6 feet.</td>
<td>−3.6 feet</td>
</tr>
<tr>
<td>A water level less than −2(\frac{1}{2}) feet</td>
<td>More than 2(\frac{1}{2}) feet below a full pool</td>
<td>−3 feet</td>
</tr>
<tr>
<td>A temperature less than −5°F</td>
<td>Colder than 5°F below 0</td>
<td>−8°F</td>
</tr>
<tr>
<td>An account balance less than −$100</td>
<td>A debt greater than $100</td>
<td>−$110</td>
</tr>
</tbody>
</table>

Absolute value equations can be used to calculate the distance between positive and negative numbers to solve real-world problems.
Integers are the set of whole numbers with their opposites. The integers can be represented by the set \{ \ldots , -3, -2, -1, 0, 1, 2, 3, \ldots \}. The three periods before and after the numbers in the set are called an ellipsis, and they are used to represent infinity in a number set.

Rational numbers are the set of numbers that can be written as \( \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \) does not equal 0.

There are many ways you can classify numbers. Many of the classifications are subsets of other classifications. The diagram shows the different sets of numbers you have encountered in your mathematical experiences.

For example, the Top Notch company’s bank balances are shown. The table shown represents the first 10 weeks of operation. Overdrafts, which are a negative balance, are represented by amounts within parentheses. What was the gain or loss between Weeks 2 and 3?

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance</td>
<td>$159.25</td>
<td>($201.35)</td>
<td>$231.57</td>
<td>($456.45)</td>
<td>($156)</td>
<td>($12.05)</td>
<td>$281.34</td>
<td>$175</td>
<td>$192.34</td>
<td>$213</td>
</tr>
</tbody>
</table>

At the end of Week 2, the company had a negative balance of $201.35 and at the end of Week 3 it had a positive balance of $231.57. The company had a gain between these two weeks because it went from a lesser balance to a greater balance. The gain is equal to the sum of the absolute values of the two balances.

\[ |-$201.35| + |$231.57| = 201.35 + 231.57 = 432.92 \]
Natural numbers are a subset of whole numbers.
Whole numbers are a subset of integers.
Integers are a subset of rational numbers.
The number 2 is a rational number, an integer, a whole number, and a natural number.
The number 0 is a rational number, an integer, and a whole number.
The number −11 is a rational number and an integer.
The numbers 12.5 and $\frac{2}{3}$ are both rational numbers.

The **Density Property** states that between any two rational numbers there is another rational number.

For example, consider the rational numbers −0.42 and −0.43 and the number line shown. The number represented by point A is another rational number that falls between −0.42 and −0.43 such that $−0.43 < A < −0.42$. Point A could represent the value −0.425.

The property is not true for natural numbers, whole numbers, or integers. For example, there is no integer between −25 and −26. There is no whole number or natural number between 12 and 13.
Air traffic controllers use radar to track tens of thousands of commercial airline flights every day. Controllers use quadrants to identify the locations, altitudes, and speeds of the many different flights.

Lesson 1
Four Is Better Than One
Extending the Coordinate Plane ............................................. M4-57

Lesson 2
It’s a Bird, It’s a Plane… It’s a Polygon on the Plane!
Graphing Geometric Figures ..................................................... M4-73

Lesson 3
There Are Many Paths...
Problem Solving on the Coordinate Plane ..................................... M4-87
TOPIC 2: THE FOUR QUADRANTS

In this topic, students explore the four quadrant coordinate plane. They use reflections of the first quadrant on patty paper and their knowledge of the rational number line to build their own four quadrant coordinate plane. Students look for patterns in the signs of the ordered pairs in each quadrant and the ordered pairs that lie along the vertical and horizontal axes. After developing a strong foundation for plotting points and determining distances on the coordinate plane, students analyze and solve problems involving geometric shapes on the coordinate plane. They use the knowledge gained throughout the course to solve a wide range of problems on the coordinate plane, using scenarios, graphs, equations, and tables.

Where have we been?
Prior to grade 6, students represented real-world and mathematical problems in the first quadrant of a coordinate plane and interpreted the coordinate values of points. In the previous topic, students extended the rational number line to include negative values. The opening activities of this topic access all of this prior knowledge as students construct the four quadrant coordinate plane.

Where are we going?
This topic provides students with an introduction to the entire real number coordinate plane. Throughout the rest of this course and in the coming years, students will represent relationships on the coordinate plane and interpret the meanings of points, lines, and other graph elements plotted on the plane. This topic provides the foundation for those lessons.

The Four-Quadrant Coordinate Plane

The intersection of a horizontal x-axis and vertical y-axis at a point called the origin divides an infinite flat plane into four quadrants.
Myth: "I’m not smart."

The word “smart” is tricky because it means different things to different people. For example, would you say a baby is “smart”? On the one hand, a baby is helpless and doesn’t know anything. But on the other hand, a baby is insanely smart because she is constantly learning new things every day.

This example is meant to demonstrate that “smart” can have two meanings. It can mean “the knowledge that you have,” or it can mean “the capacity to learn from experience.” When someone says he or she is “not smart,” are they saying they do not have much knowledge, or are they saying they lack the capacity to learn? If it’s the first definition, then none of us are smart until we acquire that information. If it’s the second definition, then we know that is completely untrue because everyone has the capacity to grow as a result of new experiences.

So, if your student doesn’t think that they are smart, encourage them to be patient. They have the capacity to learn new facts and skills. It might not be easy, and it will take some time and effort. But the brain is automatically wired to learn. Smart should not refer only to how much knowledge you currently have.

#mathmythbusted

Talking Points

You can further support your student’s learning by asking questions about the work they do in class or at home. Your student is learning to use an expanded number system in different contexts and with different graphical representations.

Questions to Ask

• How does this problem look like something you did in class?
• Can you show me the strategy you used to solve this problem? Do you know another way to solve it?
• Does your answer make sense? Why?

Key Terms

quadrants
The four regions on the coordinate plane are called quadrants. They are numbered with Roman numerals from one to four (I, II, III, IV) starting in the upper right-hand quadrant and moving counterclockwise.

ordered pairs
An ordered pair is a pair of numbers that can be represented as (x, y) to indicate the position of a point on the coordinate plane. For example, the ordered pair for the origin is (0, 0).
WARM UP
Plot each point.

A (3, 5)  B (0, 4)  C (6, 1)  D (8, 0)  E (0, 0)

LEARNING GOALS
- Identify the four quadrants of the coordinate plane and the characteristics of points located in each.
- Locate and plot ordered pairs of positive and negative rational numbers on the coordinate plane.
- Determine the relationship between the signs of coordinates of ordered pairs that are reflections across one or both axes.
- Use absolute value to determine distances on the coordinate plane.
- Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane.

KEY TERM
- quadrants

You can locate and plot ordered pairs of positive numbers on a coordinate plane. How can you extend the plane to include ordered pairs of any rational numbers?
Getting Started

All About Extending

Consider the coordinate plane that you have used to graph points where both the x- and y-coordinates were zero or positive numbers.

1. Based on what you have learned about number lines:
   a. What do you know about the number line that makes up the x-axis? Extend that number line and label it appropriately.
   b. What do you know about the number line that makes up the y-axis? Extend that number line and label it appropriately.

2. The point where the x-axis and y-axis intersect is known as the origin. Label the point of intersection with its coordinates.

   By extending the number lines that form the axes, you have created the entire coordinate plane.

3. How many regions are created when the coordinate plane is extended to all rational numbers?

   The regions on the coordinate plane are called quadrants. They are numbered with Roman numerals from one to four (I, II, III, IV) starting in the upper right-hand quadrant and moving counterclockwise.

4. Label each of the quadrants on your coordinate plane.
Your teacher is going to direct students to stand at certain locations on the human coordinate plane.

1. For each student, plot and label the point where the student is standing on the coordinate plane. Then record the coordinates of that point in the table.

<table>
<thead>
<tr>
<th>Student</th>
<th>Location</th>
<th>Student</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>H</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>I</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>J</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>K</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td>L</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td></td>
<td>M</td>
<td></td>
</tr>
<tr>
<td>G</td>
<td></td>
<td>N</td>
<td></td>
</tr>
</tbody>
</table>

2. Where did each student always start? How did each student know which direction to go first?

3. What do you notice about the coordinates of the points that are in the same quadrant of the coordinate plane?
Your teacher is going to select students to plot ordered pairs that meet specific conditions. The students will select locations that satisfy those conditions.

4. For each student, plot and label the point where the student is standing on the coordinate plane. Then record the coordinates of that point in the table.

<table>
<thead>
<tr>
<th>Student</th>
<th>Condition</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Anywhere</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Negative x-coordinate</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Negative y-coordinate</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>On an axis</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>In QII</td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>In QIII</td>
<td></td>
</tr>
</tbody>
</table>

5. Compare the ordered pairs you have plotted and identified in this activity. What is similar about the points you graphed in each region or axis of the graph?

a. QI: 

c. QIII: 

e. x-axis: 

b. QII: 

d. QIV: 

f. y-axis: 

M4-60 • TOPIC 2: The Four Quadrants
Investigating Reflections

In this activity, you will use patty paper to search for specific patterns on the coordinate plane.

Reflecting across the x-axis: Place a sheet of patty paper over the coordinate plane and trace the axes.

1. For each ordered pair,
   - Plot and label the point on patty paper.
   - Fold the patty paper on the x-axis.
   - Trace the point through the patty paper.
   - Label the coordinates of the new point.
   a. A (4, 1) \[ A' (\_\_, \_\_) \]
   b. B (−3, 4) \[ B' (\_\_, \_\_) \]
   c. C (5, −2) \[ C' (\_\_, \_\_) \]
   d. D (0, −7) \[ D' (\_\_, \_\_) \]

2. What did you notice about the coordinates of the original points and their reflections? Write a generalization for how the coordinates of a point and its reflection across the x-axis are related.
Now let’s investigate reflecting across the $y$-axis. Place a new sheet of patty paper over the coordinate plane and trace the axes.

3. For each ordered pair,
   - Plot and label the point on patty paper.
   - Fold the patty paper on the $y$-axis.
   - Trace the point through the patty paper.
   - Label the coordinates of the new point.

   a. $A (4, 1)$  
      $A'$ (____, ____)

   b. $B (-3, 4)$  
      $B'$ (____, ____)

   c. $C (5, -2)$  
      $C'$ (____, ____)

   d. $D (-3, 0)$  
      $D'$ (____, ____)

4. What did you notice about the coordinates of the original points and their reflections? Write a generalization for how the coordinates of a point and its reflection across the $y$-axis are related.
Your teacher is going to select students to plot ordered pairs that meet specific conditions. The students will select locations that satisfy those conditions.

5. For each student, plot and label the point where the student is standing on the coordinate plane. Then record the coordinates of that point in the table.

<table>
<thead>
<tr>
<th>Student</th>
<th>Condition</th>
<th>Location</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Quadrant II</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Reflection of A across the x-axis</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Reflection of B across the y-axis</td>
<td></td>
</tr>
</tbody>
</table>

6. Compare the ordered pairs for A and C. What do you notice about their coordinates? Write a generalization for how the coordinates of a point and its reflection across both axes are related.
7. For each pair of conditions, plot and label two points. Record the coordinates of the points.

a. One point is in Quadrant II. The two points are reflections of each other across the x-axis.

b. One point is in Quadrant III. The points are reflections of each other across the y-axis.

c. One point is in Quadrant IV. The points are reflections of each other across both axes.

8. In general, how are points that are reflections across one or both axes similar to and different from each other?
1. Consider points $A$ and $B$.
   a. Use the coordinate plane to determine the distance from point $A$ to point $B$.
   
   b. Describe how the coordinates of points $A$ and $B$ are similar.
   
   c. Write an absolute value equation using the $x$-coordinates of the points to calculate the distance.

2. Consider points $B$ and $F$.
   a. Use the coordinate plane to determine the distance from point $B$ to point $F$.
   
   b. Describe how the coordinates of points $B$ and $F$ are similar.
   
   c. Write an absolute value equation using the $y$-coordinates of the points to calculate the distance.

3. Write an absolute value equation and calculate the distance from:
   a. point $D$ to $(-3, -5)$.  
   b. $(-7, -4)$ to $(3, -4)$.
   
   c. $(6, 2)$ to $(6, -5)$.  
   d. point $B$ to $(-9, 2)$.
   
   e. $(8, -7)$ to point $F$. 

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**Activity 1.3**

**Horizontal and Vertical Distance on the Coordinate Plane**

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LESSON 1: Four Is Better Than One  •  M4-65
In the T-Rex Dig game, players place the “bones” of their dinosaur horizontally or vertically on a coordinate grid. Players then take turns guessing the location of each other’s dino bones using coordinates. Once a player has located all of the other player’s dino bones, the game is over.

Let’s look at a sample game board and questions that might be asked to uncover all the dino bones.

1. Use the game board to answer questions about the T-Rex fossils. (Each grid line is 1 foot long.)

![Game Board Image]

a. How long is the T-Rex’s skull? Write an absolute value equation to justify your answer.
b. How many coordinates must be guessed to completely “uncover” the skull?

c. How long is the T-Rex’s femur? Write an absolute value equation to justify your answer.

d. What is the greatest number of quadrants crossed by any one fossil?

e. Are any fossils on an axis? If so, identify the axis, the fossil, and the coordinates of the fossil(s).

2. Your turn! Use the graph paper provided at the end of the lesson. Use the bottom grid to plot and label your 5 fossils. You may want to label some of the coordinates to help you as you play the game. Use the top grid to record the coordinates you ask of your partner.

As you play the game ask your opponent mathematical questions. For example, you can ask:

- Is the femur symmetric over an axis?
- How many of your fossils are vertical?
- Are any of the fossils on an axis? (But you can’t ask which axis!)
- Do any of the fossils share an ending x-coordinate with another fossil?
TALK the TALK

Determining Coordinates

Use the graph and information provided to answer each question.

• The graph shows the locations of point F and point G.

• Point G is on the x-axis and has the same x-coordinate as point F.

• Point H is located at (−4, a).

• The distance from point F to point G is half the distance from point F to point H.

1. What is the value of \( a \)? Explain how you determined this coordinate.

2. Plot point J so that the distance from point F to point J is the same as the distance from point F to point H. Explain how you decided where to plot point J.
Assignment

Write
Use the terms *axis*, *quadrant*, and *coordinates* to explain how ordered pairs that differ only by sign are related to each other.

Remember
The Cartesian coordinate plane is formed by two perpendicular number lines that intersect at the zeros, or the origin. The intersecting number lines divide the plane into four regions, called quadrants.

Practice
1. Identify the ordered pair associated with each point graphed on the coordinate plane.

   ![Coordinate Plane Diagram]

2. Plot and label the locations of points P through Z on a coordinate plane. Draw line segments from point to point, beginning and ending at point P. Describe the resulting figure.

   - P (0, 5)
   - Q (1, 3)
   - R (4, 3)
   - S (2, 1)
   - T (4, 2)
   - V (0, -1)
   - W (-4, -3)
   - X (-2, 1)
   - Y (-4, 3)
   - Z (-1, 3)

3. Plot the ordered pair \((a, b)\) in Quadrant I of a coordinate plane and the ordered pair \((c, d)\) in Quadrant III. Plot and label each additional ordered pair. Explain how you knew where to plot each point.

   - a. \((-a, b)\)
   - b. \((a, -b)\)
   - c. \((-a, -b)\)
   - d. \((-c, d)\)
   - e. \((c, -d)\)
   - f. \((-c, -d)\)

4. The coordinate plane shown represents a map of Paul’s neighborhood. Each square represents one city block. Paul’s house is located at point A, which is the origin. The other points represent the following locations.

   - B – USA Bank
   - C – Paul’s friend Franco’s house
   - D – Gray’s Grocery Store
   - E – Post Office
   - F – Edward Middle School
   - G – Playground
   - H – Smiles Orthodontics

5. Explain how Paul can get to the given destination from his house if he were to first walk east or west and then walk north or south. Then, determine the coordinates of the destination point and the quadrant in which the point is located.

   - a. USA Bank
   - b. Smiles Orthodontics
   - c. Franco’s house
   - d. Playground
   - e. Post Office
6. Identify the ordered pairs associated with B and E. Describe how the ordered pairs are similar.
7. Write an absolute value equation using the y-coordinates of the points to calculate the distance between B and E.
8. How can an absolute value equation help you calculate the distance from one point to another on the coordinate plane when the points are on the same vertical or horizontal line?

**Stretch**
Create a rectangle $ABCD$ on a coordinate plane that meets the following conditions:

- all four points are in different quadrants
- point A is in Quadrant II with coordinates $(-a, b)$
- the distance from point A to point B is $3a$
- the distance from point A to point D is $4b$
- neither axis is a line of symmetry in the rectangle

**Review**
Determine two rational numbers that are between the two given rational numbers.
1. 3.4 and 3.5
2. $\frac{12}{5}$ and $\frac{13}{5}$

State the opposite of each number and plot both numbers on a number line.
3. $2\frac{1}{8}$
4. $-5.97$

Calculate the area of each composite figure.
5.
6.
WARM UP
1. Draw a rectangle that is not a square.
2. Draw a rhombus that is also a rectangle.
3. Draw a trapezoid that is not a parallelogram.

LEARNING GOALS
• Plot points in all four quadrants to form polygons.
• Draw polygons in the coordinate plane using coordinates for the vertices.
• Determine the area enclosed by a polygon on the coordinate plane.
• Use coordinates to determine the length of a side joining points with the same first or second coordinate.
• Solve real-world and mathematical problems with geometric shapes in all four quadrants on the coordinate plane.

You have determined area and perimeter of common polygons. You have decomposed complex figures into simpler shapes to determine their area. You have also determined the volume of right rectangular prisms. How can you use the coordinate plane to determine the area, perimeter, and even volume of shapes and objects?
Shape Up!

Your teacher will select students to participate in the activity and provide them with conditions to plot on the Human Coordinate Plane.

1. For each student, plot and label the point where the student is standing on the coordinate plane. Use a different color for each location. Then record the coordinates of the point where the student is standing in the table.

2. What shape did your classmates form at Location 1? How can you prove that they formed the given shape?

3. Record the shape formed at Location 2. Prove that your classmates formed the shape.

4. Record the shape formed at Location 3. Prove that your classmates formed the shape.
One advantage of the Cartesian coordinate plane is that it enables mathematicians to use coordinates to analyze geometric figures.

1. Graph the points on the coordinate plane, and connect the points to form a polygon.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>-5</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>-5</td>
<td>3</td>
</tr>
</tbody>
</table>

a. Identify the polygon formed and justify your answer.

b. Determine the perimeter of the polygon.

c. Determine the area of the polygon.

2. Graph the points on the plane, and connect the points to form a polygon.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>-2</td>
<td>-3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

a. What polygon is formed? Justify your answer.

b. Determine the area of the polygon.
3. Graph the points on the plane, and connect the points to form a polygon.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>−3</td>
</tr>
<tr>
<td>−2</td>
<td>−3</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

a. What polygon is formed? Justify your answer.

b. Determine the area of the polygon.

ACTIVITY 2.2 Completing Polygons on the Plane

1. The points $A (−2, 4)$ and $B (−2, −2)$ are plotted on the coordinate plane shown.

   a. Plot and label points $C$, $D$, $E$, and $F$ so that squares $ABCD$ and $ABEF$ are formed.

   b. Determine the area of each square.

   c. Compare your squares with your classmates’ squares. Are all the squares the same or different? How do you know that the squares are drawn correctly?
2. On the coordinate plane, the line segment AB is graphed.

   a. Plot and label points C and D to form parallelogram ABCD with a height of 4 units.

   b. Determine the area of your parallelogram.

   c. Compare your parallelogram with your classmates’ parallelograms. Are all the parallelograms the same or different? How do you know that the parallelograms are drawn correctly?

3. On the coordinate plane, the points A (−3, −3) and B (4, −3) are plotted to form segment AB.

   a. Plot and label point C so that a right triangle is formed.

   b. Plot and label point D so that an acute triangle is formed.

   c. Determine the areas of your triangles.

   d. Compare your triangles with your classmates’ triangles. Are all the triangles the same or different? How do you know that the triangles are drawn correctly?
4. On the coordinate plane, points $A$ and $B$ are plotted to form segment $AB$.

   a. Plot and label two points to form trapezoid $ABCD$ with a height of 5 units. Your trapezoid should cross into at least 3 quadrants.

   b. Determine the area of your trapezoid.

   c. Compare your trapezoid with your classmates’ trapezoids. Are all the trapezoids the same or different? How do you know that the trapezoids are drawn correctly?
You have been asked to advise on the design of a playground for your local elementary school. The playground is laid out in a grid with a unit of 1 foot and a merry-go-round at the center of the playground. Your project is to determine the amount of sand needed for the fossil dig sandbox and the sand pit under the swing set.

The coordinates for the fossil pit are \((-18, -7), (-10, -7), (-18, -13),\) and \((-10, -13).\)

1. Determine the volume of the fossil pit if the pit is 0.75 feet deep.

2. If the school will fill the pit halfway up with sand, determine the volume of sand that is required.

3. Each 50-pound bag of sand holds about 0.5 cubic feet of sand. Determine the number of bags of sand needed for the fossil pit.

4. Each bag of sand costs $3.80. How much will the sand cost for the fossil pit?

The coordinates for the swing set sand pit are \((15, 2), (40, 2), (15, -8),\) and \((40, -8).\)

5. Determine the volume of the swing set sand pit if the pit is 0.5 feet deep.

6. If the school has $250 to spend on sand for the swing set sand pit, how much of it can be filled with sand?
TALK the TALK 🌸

Introduction to Coordinate Proof

1. The coordinates of a parallelogram are given. Segment $AB$ is parallel to the $x$-axis.

   a. Determine the values for $a$, $b$, $c$, and $d$, if possible.

   b. Write an expression for the length of segment $AB$.

   c. Determine the vertical height of the parallelogram.

   d. Write an expression for the area of the parallelogram.

   e. If $b = 5$, determine the values for $a$, $c$, and $d$.
      Then calculate the area of the parallelogram.
### Number of Quadrants, Polygon Names, and Area Measurements

<table>
<thead>
<tr>
<th>1 Quadrant</th>
<th>2 Quadrants</th>
<th>3 Quadrants</th>
<th>4 Quadrants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>Rectangle</td>
<td>Triangle</td>
<td>Trapezoid</td>
</tr>
<tr>
<td>Any Parallelogram</td>
<td>Any Quadrilateral</td>
<td>Any Polygon</td>
<td>Rhombus</td>
</tr>
<tr>
<td>18 square units</td>
<td>16 square units</td>
<td>20 square units</td>
<td>24 square units</td>
</tr>
<tr>
<td>30 square units</td>
<td>36 square units</td>
<td>15 square units</td>
<td>50 square units</td>
</tr>
</tbody>
</table>
**Assignment**

**Write**
Explain how to use the coordinate plane and absolute value to determine perimeter and area of geometric shapes.

**Remember**
One advantage of the Cartesian coordinate plane is that it enables mathematicians to use coordinates to analyze geometric figures. The distance between two points on a coordinate plane can be calculated by using the coordinates of the two points.

**Practice**

1. Create and analyze a trapezoid.
   a. Plot and label four points on a coordinate plane that satisfy all the conditions listed:
      - Each point is in a different quadrant.
      - The four points form a trapezoid with only one pair of parallel sides.
      - The trapezoid has a height of 9 units.
      - One base of the trapezoid has a length of 6 units.
      - The second base of the trapezoid has a length of 3 units.
      - None of the points are located on an axis.
      - The trapezoid is not symmetric to either axis.
   b. Determine the area of the trapezoid.
   c. Is it possible to create a trapezoid that satisfies the conditions but has a different area? Explain.

2. Plot and identify four points across at least 2 quadrants that form a parallelogram that is not a rectangle. Determine the area of the parallelogram.

3. Plot and identify four points across at least 3 quadrants that form a non-square rectangle. Determine the area of the rectangle.

**LESSON 2:** It’s a Bird, It’s a Plane … It’s a Polygon on the Plane! • M4-85
**Stretch**

Pick’s Theorem says that the area of a polygon that has its vertices on a lattice—a field of evenly spaced points—can be calculated as follows:

- Count the number of interior points.
- Add this to half the number of boundary points (circled).
- Subtract 1.

1. Determine the area of Figure A using Pick’s Theorem.
2. The coordinate plane can be like a lattice of points. How can you use this fact to determine the area of the given square?
3. Demonstrate Pick’s Theorem on the coordinate plane using other polygons drawn in all four quadrants.

![Figure A](image)

---

**Review**

1. Calculate the distance of each number from 125. Use positive numbers to indicate the distance when the number is greater than 125 and negative numbers to indicate the distance when the number is less than 125.
   a. 107
   b. 161
   c. 87
   d. 232

2. Graph the solution set for each given inequality.
   a. \( x > 7.75 \)
   b. \( x \leq \frac{5}{2} \)
There Are Many Paths ...
Problem Solving on the Coordinate Plane

WARM UP
Solve each equation.
1. $120 + h = 315$
2. $w - 17 = 38$
3. $\frac{c}{5} = 12$
4. $169 = 13w$

LEARNING GOALS
- Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane.
- Interpret the meaning of points plotted on the coordinate plane.
- Use equations to solve real-world problems.
- Use graphs relating an independent and dependent quantity changing in relationship to one another to solve real-world problems.
- List advantages and disadvantages of different representations for solving real-world and mathematical problems on the coordinate plane.

Now that you understand how to plot points in all four quadrants of the coordinate plane, you can solve many more types of problems than you could previously. How can you use graphs and equations to solve problems?
Getting Started

Emma’s Birthday

Analyze the graph.

1. Explain what you can determine about the situation from the graph.

2. What do the plotted points mean in terms of this situation?

3. Do all of the values on the line make sense in terms of the situation?

4. Can you determine an equation for the graph?
Julio is a wrestler for his high school team. Although he does not wrestle during the 12 weeks of summer, his coach would like him to stay around 140 pounds so that he doesn’t have to work so hard during the season to stay in his 142-pound weight class. Julio charted his weight over the summer.

<table>
<thead>
<tr>
<th>Week</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>144.5</td>
<td>142.1</td>
<td>138.5</td>
<td>139.5</td>
<td>137.5</td>
<td>141.5</td>
<td>136.25</td>
<td>137.2</td>
<td>140</td>
<td>141.3</td>
<td>138.5</td>
<td>135</td>
</tr>
<tr>
<td>Weight Differential</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Consider the table shown.

   a. Which quantity is the independent quantity and which is the dependent? Explain your reasoning.

   b. What is the unit for each quantity?

   c. Which quadrant(s) will you need in order to plot Julio’s data? Draw and label your axes. Then graph the data.
2. The coach was impressed with Julio’s data collection, but he was interested in how much Julio’s weight varied from 140 pounds each week.

a. Complete the last row of weight differentials, the differences of Julio’s weight from 140 pounds. Use negative numbers when the weight is below 140 pounds and positive numbers when his weight is above 140 pounds.

b. What is the dependent quantity in this situation?

c. Which quadrant(s) will you need in order to plot Julio’s data for the coach’s request? Draw and label your axes, including the units. Then graph the data.
3. Compare the two approaches taken by Julio and his coach.
   a. Compare the independent and dependent quantities.

   b. Compare the graphs. What do you notice about the patterns of the points?

   c. Explain the meaning of the x-axis in each approach.

   d. Why do you think the coach preferred his approach over Julio’s approach?

4. Use the table and graphs to answer each question.
   a. Between which two consecutive weeks did Julio’s weight change the most? What was the weight change?

   b. What is the difference between Julio’s highest weight and his lowest weight?

   c. Which representation—table, Julio’s graph, the coach’s graph—did you use to answer the questions? Why did you make those choices?

   d. If you were Julio’s coach, what advice would you give Julio?
An interesting day of temperature changes occurred in Rapid City, South Dakota, on January 22, 1943. The table shows the temperature changes that happened throughout the day.

<table>
<thead>
<tr>
<th>Time</th>
<th>Temperature (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10:30 A.M.</td>
<td>-6.7</td>
</tr>
<tr>
<td>10:35 A.M.</td>
<td>13.3</td>
</tr>
<tr>
<td>12:00 P.M.</td>
<td>15.6</td>
</tr>
<tr>
<td>12:05 P.M.</td>
<td>-10.6</td>
</tr>
<tr>
<td>12:35 P.M.</td>
<td>-9.4</td>
</tr>
<tr>
<td>12:40 P.M.</td>
<td>10</td>
</tr>
<tr>
<td>2:20 P.M.</td>
<td>14.4</td>
</tr>
<tr>
<td>2:25 P.M.</td>
<td>-8.3</td>
</tr>
</tbody>
</table>

Create a graph of the temperature changes.

1. Which quadrants do you need for your graph? Explain your reasoning.

2. Draw and label the axes for the graph. Then graph the data and connect consecutive points.
3. Between which two times was the temperature swing the greatest?

4. Describe the pattern. Why is this called an “interesting” day?
Suppose this graph summarizes your day. The x-axis of this graph represents time in minutes from 12:00 P.M., and the y-axis represents your distance from home in blocks. Locations north of your house are positive, and locations south of your house are negative. A point at the origin represents you being home at 12:00 P.M.

1. Describe the meaning of each of the four labeled points.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

2. Adrian and Sierra are discussing how the graph should look before $x = -6$ and after $x = 10$. Adrian thinks he should draw arrows to indicate that the graph continues to the left and right, respectively. Sierra disagrees and thinks they should draw segments back to the x-axis. Who is correct?
Let’s consider another graph.

Let’s consider another graph.

3. Write a possible scenario for this graph. Be sure to specify the units and the meaning of the origin for your scenario.

Natasha and her family took a 3-day trip to her grandmother’s house. On the first day, they drove 300 miles. On the second day, they drove 350 miles. On the third day, they drove the remaining 200 miles.

4. Create a graph to represent Natasha’s family trip. Be sure to label your axes with quantities and units and label specific points that highlight the trip.
Nadja is coordinating the neighborhood Spring Fling. She asks Matthew to blow up balloons for the event. The graphs shown represent his efforts.

5. Analyze each graph shown, and then answer each question.

a. What quantity is represented on the x-axis in each graph?

b. What quantity is represented on the y-axis in each graph?

c. Which quantity is independent quantity and which is dependent quantity?
6. Match each description with the appropriate graph.

a. Matthew blows air into a balloon at a steady rate, then ties it off when it is full.

b. Matthew blows air into a balloon, and then the balloon pops!

c. Matthew blows air into a balloon, and then lets the air out.

d. Matthew blows air into a balloon slowly. As the balloon stretches out, he is able to blow more air into the balloon. He then ties off the balloon when it is full.
The graph shows the water level of a pool. The x-axis represents time, in hours, and the y-axis represents the water level, in inches. The origin represents 3:00 P.M. and the desired water level.

1. Label the graph with the independent and dependent quantities and their units.

2. Create a table of values for the points plotted and describe the meaning of each.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. At what rate did the water go into the pool? Explain your reasoning.

4. Describe a situation that would match the graph.

5. Write an equation for this situation.

6. Why does the graph stop rather than continue infinitely?

7. Using any of your mathematical tools, determine the time when the pool was 3 inches above the desired fill level. Is your answer exact or approximate? Explain.
As part of a science project, Damon collected water in a bucket in his backyard and is studying the evaporation. Unfortunately, Damon is a bit forgetful and forgets to take measurements of the water every day. The first day he remembered was Sunday, which was 4 days AFTER the data collection was to begin. He collects the following data.

<table>
<thead>
<tr>
<th>Days Since Sunday</th>
<th>Height of Water (inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
</tr>
<tr>
<td>12</td>
<td>15</td>
</tr>
</tbody>
</table>

1. Graph the data. Connect the data values with a line. Be sure to label your axes.
2. Assuming that the water evaporated at the same rate every day, use your graph to determine the water level the day he was supposed to start data collection.

3. Assuming that the water evaporated at the same rate every day, use your graph to determine when the water level was
   a. 30 inches.
   b. 12 inches.
   c. 5 inches.
   d. 0 inches.

4. Explain why you should or should not extend your graph into Quadrant IV.
Your friend Aidan got a job working at the local hardware store. He created the graph shown to track how much money he makes for a given number of hours.

1. Create a table of values for Aidan’s graph.

<table>
<thead>
<tr>
<th>Hours Worked</th>
<th>Pay ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

2. How can you tell, by looking at the graph, whether the graph displays equivalent ratios? If it does, what is the ratio, or rate, displayed in the graph?
3. Define variables for the hours worked and Aidan’s pay.

4. Write an equation to describe Aidan’s graph.

5. Use the tool of your choice—equation, graph, or a table—to answer each equation.
   
   a. Approximately how much money did Aidan make if he worked 15 hours this week?
   
   b. Determine the exact amount of money Aidan made if he worked 12 hours this week.
   
   c. Approximately how many hours did Aidan work if he made $50 this week?
   
   d. Determine the exact number of hours Aidan worked if he made $152.50 this week.
   
   e. How did you decide which tool to use to answer each question?
Jason and Liliana need to measure some pictures so they can buy picture frames. They looked for something to use to measure the pictures, but could find only a broken yardstick. The yardstick was missing the first $2 \frac{1}{2}$ inches.

They both thought about how to use this yardstick.

Lilianna said that all they had to do was measure the pictures and then subtract $2 \frac{1}{2}$ inches from each measurement.

1. Is Lilianna correct? Explain your reasoning.

2. They measured the first picture’s length and width to be 11 inches and $9 \frac{1}{2}$ inches. What are the actual length and width?

3. Define variables for a measurement with the broken yardstick and the actual measurement.

4. Write an equation that models the relationship between the variables.
5. Complete the table of values for the measurement on the yardstick and the actual measurement.

<table>
<thead>
<tr>
<th>Measurement with Broken Yardstick (in.)</th>
<th>Actual Measurement (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>(9\frac{1}{2})</td>
<td></td>
</tr>
<tr>
<td>(25\frac{3}{4})</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td></td>
</tr>
<tr>
<td>(18\frac{5}{8})</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
<tr>
<td>(29\frac{1}{8})</td>
<td></td>
</tr>
<tr>
<td>(6\frac{7}{8})</td>
<td></td>
</tr>
</tbody>
</table>

6. Use the table to complete the graph of the actual measurements versus the measurement taken with the broken yardstick.

7. Would it make sense to connect the points on this graph? Explain why or why not.
8. Suppose the yardstick was broken at 5 inches instead of $\frac{21}{2}$ inches.

   a. Write the new equation for the relationship between the actual measurement and the measurement from the broken yardstick.

   b. Sketch a graph of the actual measurements versus the measurement taken with the new broken yardstick on the graph with the original yardstick.

   c. What do you notice about the two graphs?

   d. What is the meaning of the $x$-intercept—the point with a $y$-coordinate of zero—on each graph?
A freediver is a person who dives into the ocean without the use of any breathing device like scuba equipment. William Trubridge holds the record for freediving. In 2016, he broke his own record and dove almost 407 feet into the ocean! Suppose you plan to train as a freediver and want to beat Trubridge’s record.

1. What are some questions you would ask of Trubridge about his dive?

2. Assume that Trubridge ascended and descended at the same rate of 2.97 feet per second to help you determine how much time you need to be able to hold your breath to beat Trubridge’s record.
TALK the TALK

Your Turn!

You and your group should prepare a presentation for this problem.

1. Create a situation that can be modeled by the graph.

Write at least 3 sentences for what you want to say during your presentation.

- Be sure to determine the ratio, or rate, for how the variables change in relation to each other.
- Describe the meaning of each point on the graph.
- Define variables for the independent and dependent quantities based on your situation.
- Write an equation to represent the problem situation.
**Practice**

1. The gravitational pull of the Moon is not as great as that of Earth. In fact, if a person checks his weight on the Moon, it will be only $\frac{1}{6}$ of his weight on Earth.
   a. If a person weighs 186 pounds on Earth, how much will he weigh on the Moon? How many pounds different from his actual weight is that?
   b. Complete the table of values for a person’s weight on Earth, weight on the Moon, and difference of the two weights. Use negative numbers when the weight is less than the person’s earth weight.

<table>
<thead>
<tr>
<th>Weight on Earth (lb)</th>
<th>186</th>
<th>168</th>
<th>198</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight on Moon (lb)</td>
<td></td>
<td>29</td>
<td>21</td>
</tr>
<tr>
<td>Weight Differential</td>
<td>–155</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Graph the weight differential versus the weight on Earth. Be sure to label your axes.

2. To keep her students relaxed and focused during tests, Ms. Chappell puts small bowls of candy on each of their desks. Write a short story to describe each graph.
   a. [Graph a](#)
   b. [Graph b](#)
   c. [Graph c](#)
   d. [Graph d](#)
3. The following graph shows the average temperature, in degrees Celsius, in Fairbanks, Alaska. The x-axis represents time in days from January 1, and the y-axis represents degrees Celsius.

![Graph](image)

a. Label the axes with the independent and dependent quantities and their units.
b. Create a table of values for the points on the graph and describe the meaning of each.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. At what rate did the temperature increase?
d. Define variables for the quantities that are changing, and write an equation for this situation.

4. Sarina’s dog, Bruno, has to go on a diet! Sarina puts Bruno on a diet plan of daily exercise and a special type of dog food. She estimates Bruno will lose $\frac{3}{2}$ pounds per week on this plan.

a. How many pounds does Sarina estimate Bruno will lose in 2 weeks? In $8 \frac{1}{2}$ weeks?
b. Define variables for the independent and dependent quantities for this situation.
c. Write an equation for this situation. (Because Bruno is losing weight, the number of pounds he loses will be defined as a negative value.)
d. Create a table of values for the situation.
e. Complete a graph of the situation.
f. Explain what points in Quadrant I would mean for Bruno.
### Stretch
Tell a story to describe the graph.

![Graph](image)

### Review
1. The vertices of a polygon are given. Plot the points on a coordinate plane and connect the points in the order they are listed. Then determine the area of the polygon.
   
   $$(-4, -1), (-3, -2), (10, -2), (3, 0), (0, 4), (-2, 3)$$

2. Create a scenario to fit each numeric expression.
   
   a. $$|3 + 21|$$
   
   b. $$|8 - 3|$$

3. Evaluate each expression for the given values.
   
   a. $$5.2r + 1.2$$, when $$r = 1.5$$ and $$4.1$$
   
   b. $$\frac{1}{2}t + \frac{3}{4}$$, when $$t = \frac{2}{3}$$ and $$\frac{9}{5}$$
The Four Quadrants

Summary

KEY TERM
• quadrants

LESSON 1
Four Is Better Than One

The Cartesian coordinate plane is formed by two perpendicular number lines that intersect at the zeros, or the origin. The intersecting number lines divide the plane into four regions, called quadrants.

The quadrants are numbered with Roman numerals from one to four (I, II, III, IV) starting in the upper right-hand quadrant and moving counterclockwise.

To plot an ordered pair on the coordinate plane, begin at the origin (0, 0), and first move the distance along the x-axis given by the x-value of the ordered pair. Move right for a positive value and move left for a negative value. Then, move the distance along the y-axis given by the y-value of the ordered pair. Move up for a positive value and move down for a negative value.
For example, the following points are plotted on the coordinate plane:

- \( A (-4, 1) \)
- \( B (-1, 0) \)
- \( C (-6, -5) \)
- \( D (2, -3) \)
- \( E (0, 3) \)
- \( F (5, 3) \)

The values of the coordinates of points that are in the same quadrant will have the same sign before their \( x \)- and \( y \)-values.

Reflecting a point on the coordinate plane across the \( x \)-axis results in a new point with the same \( x \)-value and the opposite \( y \)-value as the original point.

For example, reflecting point \( A (8, 4) \) across the \( x \)-axis gives point \( A' (8, -4) \). Reflecting point \( B (-5, -9) \) across the \( x \)-axis gives point \( B' (-5, 9) \).

Reflecting a point on the coordinate plane across the \( y \)-axis results in a new point with the opposite \( x \)-value and the same \( y \)-value as the original point.

For example, reflecting point \( C (3, -2) \) across the \( y \)-axis gives point \( C' (-3, -2) \). Reflecting point \( D (-1, 0) \) across the \( y \)-axis gives point \( B' (1, 0) \).

You can use absolute value to determine distances on the coordinate plane.

For example, the distance from point \( P \) to point \( Q \) is \( |3| + |-3| = 3 + 3 = 6 \) units.

The distance from point \( P \) to point \( S \) is \( |-6| + |6| = 6 + 6 = 12 \) units.

The distance from point \( R \) to point \( S \) is \( |6| - |3| = 6 - 3 = 3 \) units.

The distance from point \( Q \) to point \( T \) is \( |7| - |-3| = 7 - 3 = 4 \) units.
One advantage of the Cartesian coordinate plane is that it enables mathematicians to use coordinates to analyze geometric figures.

For example, the points in the table have been graphed on the coordinate plane and connected to form a polygon.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>−4</td>
</tr>
<tr>
<td>1</td>
<td>−4</td>
</tr>
</tbody>
</table>

The polygon has opposite sides that are parallel and congruent, so it is a parallelogram. It also has four right angles, so it is a rectangle. The perimeter and area of the rectangle can be calculated by first determining its length and width. The length of the rectangle is 5 units and the width of the rectangle is 4 units.

Perimeter: $4 + 5 + 4 + 5 = 18$ units
Area: $5 \times 4 = 20$ square units

There is often more than one way to complete a polygon on the coordinate plane when given a segment.

For example, on the coordinate plane, the line segment $AB$ is graphed.

Plot and label points $C$ and $D$ to form a parallelogram with a height of 6 units.
Two different examples of Parallelogram $ABCD$ are shown. Each has a length of 7 units and height of 6 units, so they both have an area of $7 \times 6 = 42$ square units.

The distance between two points on a coordinate plane can be calculated by using the coordinates of the two points.

For example, the design of a playground is laid out in a grid with a unit of 1 foot. The coordinates of the sand pit that will go under the swing set are located at $(-15, 7), (-10, 7), (-15, -1),$ and $(-10, -1)$. Determine the volume of the sand pit if the pit is 0.5 foot deep.

Plotting the coordinates of the sand pit on a coordinate plane shows that the shape of the sand pit is a rectangle. Use the coordinates to determine the distance between the points which will give you the length and width of the rectangle.

- Width: $|-15| - |-10| = 15 - 10 = 5$ feet
- Length: $|7| + |-1| = 7 + 1 = 8$ feet

Area: $8 \times 5 = 40$ square feet
Volume: $8 \times 5 \times 0.5 = 20$ cubic feet
Graphs, tables, equations, and scenarios provide various information and allow for different levels of accuracy when solving problems.

For example, the graph given shows the water level of a pool. The $x$-axis represents time, in hours, and the $y$-axis represents the water level, in inches.

The origin represents 3:00 P.M. and the desired water level.

You can create a table of values for the points plotted and describe the meaning of each.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-8$</td>
<td>$-6$</td>
<td>At 7:00 A.M., the water level is 6 inches below the desired water level.</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-1\frac{1}{2}$</td>
<td>At 1:00 P.M., the water level is $1\frac{1}{2}$ inches below the desired water level.</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>At 3:00 P.M., the water is at the desired water level.</td>
</tr>
<tr>
<td>$4$</td>
<td>$3$</td>
<td>At 7:00 P.M., the water level is 4 inches above the desired water level.</td>
</tr>
<tr>
<td>$8$</td>
<td>$6$</td>
<td>At 11:00 P.M., the water level is 6 inches above the desired water level.</td>
</tr>
</tbody>
</table>

You can use the graph to determine that the water went into the pool at a rate of $\frac{3}{4}$ inch per hour. An equation that represents this situation would be $y = \frac{3}{4}x$. 

**TOPIC 2: SUMMARY**
The lessons in this module build on the data displays that you have used in elementary school, namely line plots, bar graphs, and circle graphs. You will be introduced to the field of statistics, the study of data, and the statistical problem-solving process. You will calculate numerical summaries to describe a data set. You will also learn what separates mathematical and statistical reasoning—the presence of variability.

**Topic 1  The Statistical Process ................................. M5-3**

**Topic 2  Numerical Summaries of Data ........................ M5-67**
On average, one out of every 25 sheep has black wool. A quick way to estimate the size of a flock of sheep is to count the black sheep and multiply by 25.

**Lesson 1**

*What's Your Question?*

Understanding the Statistical Process ........................................ M5-7

**Lesson 2**

*Get in Shape*

Analyzing Numerical Data Displays ............................................. M5-25

**Lesson 3**

*Skyscrapers*

Using Histograms to Display Data .............................................. M5-47
TOPIC 1: THE STATISTICAL PROCESS
In this topic, students are introduced to the statistical problem-solving process: formulate questions, collect data, analyze data, and interpret the results. Students will use this process throughout their studies of statistics, increasing the complexity of each step of the process as they develop their statistical literacy. Students use bar graphs and pie charts to analyze and interpret survey data, the final steps of the statistical process. As students learn about and analyze dot plots, stem-and-leaf plots, and histograms, they practice using the four steps of the statistical process.

Where have we been?
In grade 1, students were expected to organize, represent, and interpret data with up to three categories. In grades 2 and 3, students created picture graphs and bar graphs of categorical data. In grades 4 and 5, students made line plots to display data with fractions. And in grade 4, students developed a conceptual understanding of angles and angle measurement, allowing them to create pie charts.

Where are we going?
In grade 7, students will use the data displays learned in this topic to compare data distributions. They will use statistical problem solving to investigate and draw inferences about populations. In grade 8, students will move into comparing two variables with data.

Histograms
A histogram is a graphical way to display quantitative or numerical data using vertical bars. The width of a bar in a histogram represents an interval of data and is often referred to as a bin. The height of the bar indicates the frequency, or the number of data values included in any given bin. Bins are represented by intervals of data instead of showing individual data values.
Myth: Faster = smarter.

In most cases, speed has nothing to do with how smart you are. Why is that? Because it largely depends on how familiar you are with a topic. For example, a bike mechanic can look at a bike for about 8 seconds and tell you details about the bike that you probably didn’t even notice (e.g., the front tire is on backwards). Is that person smart? Sure! Suppose, instead, you show the same bike mechanic a car. Will they be able to report the same amount of detail as they did for the bike? No!

It’s easy to confuse speed with understanding. Speed is associated with the memorization of facts. Understanding, on the other hand, is a methodical, time-consuming process. Understanding is the result of asking lots of questions and seeing connections between different ideas. Many mathematicians who won the Fields Medal (i.e., the Nobel prize for mathematics) describe themselves as extremely slow thinkers. That’s because mathematical thinking requires understanding over memorization.

#mathmythbusted

Talking Points
You can support your student’s learning by approaching problems slowly. Students may observe a classmate learning things very quickly, and they can easily come to believe that mathematics is about getting the right answer as quickly as possible. When this doesn’t happen for them, future encounters with math can raise anxiety, making problem solving more difficult, and reinforcing a student’s view of himself or herself as “not good at math.” Slowing down is not the ultimate cure for math difficulties. But it’s a good first step for children who are struggling. You can reinforce the view that learning with understanding takes time, and that slow, deliberate work is the rule, not the exception.

Key Terms

categorical data
Categorical data are data for which each piece of data fits into exactly one of several different groups or categories. Categorical data are also called qualitative data.

quantitative data
Quantitative data are data for which each piece of data can be placed on a numerical scale. Quantitative data are also called numerical data.

frequency
Frequency is the number of times an item or number occurs in a data set.

mode
The mode is the value or values that occur most frequently in a data set.
LESSON 1: What's Your Question?   •   M5-7

LEARNING GOALS
• Recognize and design statistical questions and anticipate variability in data related to the question.
• Differentiate between surveys, observational studies, and experiments.
• Describe the four stages of the statistical process.
• Discuss the different types of data that can be collected, displayed, and analyzed.
• Analyze and interpret bar graphs and circle graphs.

KEY TERMS
• variability
• data
• statistical question
• statistical process
• categorical data
• quantitative data
• population
• sample
• survey
• observational study
• experiment
• bar graph
• circle graph
• frequency
• mode

WARM UP
Ms. White asked the 25 sixth graders in her class, “How many pets do typical 6th graders in our class have?” Ms. White summarized the responses in the table.

<table>
<thead>
<tr>
<th>Number of Students</th>
<th>Number of Pets</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>38</td>
</tr>
</tbody>
</table>

1. About how many pets does each sixth grader in Ms. White’s class own? How did you make your decision?

You have been solving mathematical problems throughout this course. Now, you are going to study statistical problems. How are mathematics and statistics similar and different?
Statistical or Not, That Is the Question

Have you ever wondered, “How much money do professional athletes make?” Or, “How long are the books assigned to sixth graders?” If so, you have asked a statistical question. If you have sought out the answer to your question, you have engaged in the statistical process.

Cut out the questions provided at the end of the lesson. Read each question and sort them into as many groups as you would like. There must be more than one group and there must be at least two questions per group.

1. Record your groups and the questions in each group.

In this module, you will begin your formal study of statistics and the statistical process. Statistics is a problem-solving process, also called an investigative process, because the heart of statistics is about determining a possible answer to a question that has variability.

Statistical problem solving begins with a statistical question. A statistical question is a question that anticipates an answer based on data that vary.

2. Which questions from your sort are statistical questions? Explain how you would expect the answers to those questions to vary.
The **statistical process** has four components:

- Formulating a statistical question.
- Collecting appropriate data.
- Analyzing the data graphically and numerically.
- Interpreting the results of the analysis.

This lesson provides an overview of the statistical process, but you will continue to use the process throughout your study of statistics. Statistics is about posing interesting questions that you want to answer about varying attributes.

Analyze the questions posed by Bianca and Rajan.

**Bianca**

“What clubs am I in?”
“How many students are in the Chess Club?”

**Rajan**

“What clubs do my classmates belong to?”
“How many members do the clubs at my school have?”

1. **Explain why Bianca’s questions are not statistical questions but Rajan’s are.**

2. **What kinds of answers do you expect from Rajan’s questions?**
Answering a statistical question requires collecting variable data. You will learn about two types of data: categorical data and quantitative data.

3. Would the answers to Rajan’s questions be categorical or quantitative?

4. Gather the statistical questions from the Statistical or Not activity. Which questions have categorical answers and which have quantitative answers?

5. For each question, determine if it is a statistical question. If it is not, rewrite it as a statistical question. Then, state if the data would be categorical or quantitative.

   a. How many text messages did you send and receive yesterday?
   b. What are the most popular school mascots?
   c. How much time did you spend watching TV or playing video games last weekend?

Categorical data, or qualitative data, are data for which each piece of data fits into exactly one of several different groups or categories.

Quantitative data, or numerical data, are data for which each piece of data can be placed on a numerical scale and compared.
d. How many hours do 6th graders sleep each night?

e. What is your favorite sport?

6. Write at least 2 additional statistical questions that you would be interested in answering. State if the data would be categorical or quantitative.

Some data, like area codes, are numbers but are not quantitative variables. This data serves as a label, or category.
For this activity, let’s consider the topic of school lunches.

1. Write three statistical questions that you can ask about school lunches.

   a. 

   b. 

   c. 

The second component of the statistical process is to collect the data to answer the statistical question.

A statistical question can be answered by collecting data from an entire population or, more commonly, from a sample of the population. A population is an entire set of items from which data are collected. A sample is a selection from a population.

For example, to answer the question “How tall are 6th graders?” using the population of all 6th graders, you would need to determine the heights of every 6th grader in the world. However, you could choose to answer the question by collecting data from a sample of 6th graders—the 6th graders at your school.

Three common methods of data collection are surveys, observational studies, and experiments. In a survey, people are asked one or more questions. Similarly, in an observational study, the researcher (you!) collects data by observing the variable of interest. In an experiment, the researcher imposes a condition and observes the results.
You could conduct an experiment to investigate if 6th graders perform better on an assessment if they read a textbook or watch a video about the material. You would randomly assign half the students to read the text and half the students to watch the video. All students would be given the same assessment. You would compare the scores of the students in the two groups.

2. For each statistical question you wrote in Question 1, identify the population and sample of interest.
   a. 
   b. 
   c. 

3. Do you think a survey, observational study, or experiment would be the best way to collect the data to answer your statistical questions? Explain your reasoning.
   a. 
   b. 
   c.
Suppose you are interested in characteristics of sixth graders at your school.

1. **Formulate three categorical statistical questions and survey your class to obtain a sample.**

In the statistical process, after you collect the data, it is time to analyze and interpret the results. Analysis includes selecting the most appropriate graphical display and numerical summaries for your question and your method of data collection.

You already have experience displaying and summarizing categorical data using *bar graphs* and *circle graphs*.

A **bar graph** displays categorical data using either horizontal or vertical bars on a graph. The height or length of each bar indicates the value for that category.

A **circle graph**, often called a pie chart, displays categorical data using sectors, or “wedges,” of a circle. It shows how parts of the whole relate to the whole and how parts of the whole relate to the other parts. The area of each sector corresponds to the percentage of the part in relation to the whole.
Nicole and Neal were interested in the favorite sports of 6th graders. They surveyed their class of 30 students. Then, they displayed their class's data in different ways. Analyze each graph.

2. How are the graphs similar? How are they different?
In order to create the graphs, Nicole and Neal determined the frequency of each response and recorded the frequencies in a frequency table. A frequency is the number of times an item or number occurs in a data set. Once the frequency is known, you can determine the mode. The mode is the value or values that occur most frequently in a data set.

<table>
<thead>
<tr>
<th>Sport</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Football</td>
<td>4</td>
</tr>
<tr>
<td>Softball</td>
<td>3</td>
</tr>
<tr>
<td>Basketball</td>
<td>7</td>
</tr>
<tr>
<td>Baseball</td>
<td>2</td>
</tr>
<tr>
<td>Wrestling</td>
<td>1</td>
</tr>
<tr>
<td>Gymnastics</td>
<td>5</td>
</tr>
<tr>
<td>Volleyball</td>
<td>3</td>
</tr>
<tr>
<td>Track</td>
<td>3</td>
</tr>
<tr>
<td>Swimming</td>
<td>2</td>
</tr>
</tbody>
</table>

3. What conclusions can you make about the most popular sport in Nicole and Neal’s class? Use the table and graphs to explain your reasoning.
4. Compile your class’s responses to the 3 survey questions you asked in Question 1. Record the frequency of each response in a table.

5. Create a graphical display for your assigned survey question. What conclusions can you make about your class based on your graph?
TALK the TALK

(Graphically) Organizing the Process

Complete the graphic organizer for the statistical process. In each section, summarize what you know about the component and provide examples.

1. Formulate a Statistical Question
2. Collect Data
3. Analyze the Data
4. Interpret the Results
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>What is your favorite sport?</td>
<td>B</td>
</tr>
<tr>
<td>D</td>
<td>How far do I travel to school?</td>
<td>E</td>
</tr>
<tr>
<td>G</td>
<td>How many siblings do 6th graders have?</td>
<td>H</td>
</tr>
<tr>
<td>J</td>
<td>What is your favorite color?</td>
<td>K</td>
</tr>
<tr>
<td>M</td>
<td>What kinds of sports do 6th graders prefer?</td>
<td>N</td>
</tr>
</tbody>
</table>
Write
Match each definition to its corresponding term.

1. an entire set of items from which data can be selected
2. the information that is collected from an experiment, study, or survey
3. a question that anticipates variability
4. imposing a condition to test a specific result
5. a method for collecting information by asking one or more questions
6. a method for collecting information by observing a phenomenon in action
7. a subset of a population
8. the value of an attribute, or quality, being studied can change from one person or thing to another
9. data for which each piece of data fits into exactly one of several different groups or categories
10. data which can be placed on a numerical scale and compared, and can consist of discrete or continuous variables
11. a graph that shows how parts of the whole relate to the whole and how parts relate to other parts
12. a way of displaying categorical data by using either horizontal or vertical bars so that the height or length of the bars indicates the value for that category
13. the number of times an item or number occurs in a data set
14. the observation or value that occurs the most

a. data  b. experiment  c. bar graph  d. variability  e. statistical question  f. categorical data  g. circle graph  h. survey  i. observational study  j. population  k. sample  l. frequency  m. mode  n. quantitative data

Remember
There are four components to the statistical process:

- Formulate a statistical question.
- Collect data.
- Analyze the data using graphical displays and numerical summaries.
- Interpret the results in terms of the original statistical question and context.
Practice

1. Determine whether each given question is a statistical question. If not, rewrite it to make it a statistical question.
   a. How many people in your class like to play video games?
   b. Is pizza your favorite food?
   c. What time do you go to bed on school nights?

2. Determine whether a survey, observational study, or experiment would be the best way to answer each given statistical question.
   a. “How many of the students in your class ate breakfast this morning?”
   b. “Which students in your school can run a 40-meter sprint the fastest?”
   c. “How many students in your class can type at least 30 words per minute?”
   d. “How many students in your class ride the bus to school each day?”

3. Determine whether each set of given data are categorical or quantitative. If the data are quantitative, determine whether they are discrete or continuous.
   a. Each student in your math class records their height.
   b. The members of the Horse Club list the types of horses they have.
   c. The members of the Horse Club list the numbers of horses they each have.

4. Tamara claims that Sweet Grove apple juice tastes better than Juicy Bushels apple juice. Isaac claims that there is no difference between the 2 types of apple juice. Tamara and Isaac would like to find the answer to the following question: Do more 6th graders prefer Sweet Grove apple juice or Juicy Bushels apple juice?
   a. Is this a statistical question? Explain your reasoning.
   b. Explain how this question can be answered with an experiment.

5. The circle graph shows the results of the vote for the new school mascot.
   a. If 400 students voted, how many students voted for the Cheetahs?
   b. Create a bar graph to display the information, in terms of frequency, of each mascot.
   c. What conclusions about the question of what mascot should be adopted can you make based on the graphs?
Stretch
In 1945, George Polya published a book about mathematical problem solving. He outlined a four-step process for problem solving:

1. Understand the Problem
2. Devise a Plan
3. Carry out the Plan
4. Look Back

Research the four steps and explain how the four-component statistical problem-solving process is similar to and different from Polya’s four steps for mathematical problem solving.

Review
1. Choose the graph that best represents each scenario. Explain your reasoning.
   a. Carla fills a mug with tea. Every few minutes Carla takes a drink from the mug.
      A.  
      B.  
      C.  
   b. When Jamal rides his bike up a hill, his speed decreased. When he rides down a hill, his speed increased.
      A.  
      B.  
      C.  

2. Use absolute value equations to justify each answer.
   a. Determine the distance between the horizontal lines that contain points A (7, 5) and B (−4, −8).
   b. Determine the distance between the vertical lines that contain points A (7, 5) and B (−4, −8).

3. Insert a >, <, or = symbol to make each number sentence true.
   a. $−9\frac{1}{8}$  
   b. 0.006  

WARM UP
Mr. Garcia surveyed his class and asked them what types of pets they owned. Analyze the pictograph that shows the results of his survey:

Pets Owned by Students in Mr. Garcia’s Class
(each symbol represents 1 student)

Dogs 🐶 🐶 🐶 🐶 🐶 🐶 🐶 🐶 🐶 🐶
Cat 🐱 🐱 🐱 🐱 🐱 🐱 🐱 🐱 🐱 🐱 🐱
Fish 🐟 🐟 🐟 🐟 🐟 🐟 🐟 🐟 🐟 🐟
Birds 🐦 🐦
Other * * * *

1. How many students in Mr. Garcia’s class own dogs?
2. How many students own fish or birds?
3. Can you tell by looking at the pictograph how many students own pets? Why or why not?

You know how to use picture graphs, bar graphs, and line plots to display categorical and numerical data. What additional plots can be used to display and analyze numerical data?

LEARNING GOALS
• Create and interpret dot plots.
• Create and interpret stem-and-leaf plots.
• Describe the center, spread, and overall shape of a data distribution.

KEY TERMS
• dot plot
• distribution
• symmetric
• skewed right
• skewed left
• clusters
• gaps
• peaks
• outliers
• stem-and-leaf plot
Rock-Climbing Competition

Ms. Nicholson poses the question “Which grade has the fastest average rock-climbing time if each student is given one attempt?”

She selects one class from each grade level, times each student as they climb the rock wall, and records the times. Then she creates data displays for each class.

1. How are these data displays similar? How are they different?

2. What can you observe from a data display that you cannot see from looking at the numerical data?
The 2014 Winter Olympics were held in Sochi, Russia. While watching the Olympics, Jessica and Maurice decided to pose statistical questions about the Games.

1. Jessica asked, “How many medals did the United States win? How many of those were gold?” Maurice thought a better set of questions would be, “What is the typical number of medals won? What is the typical number of gold medals won by a country?” Who’s correct? Explain your reasoning.

The table at the end of the lesson lists the number of gold medals and the total medals won by all medal-winning countries for the 2014 Winter Olympics.

2. Analyze the data shown in the table.
   a. What conclusions can you make about the numbers of total medals won at the 2014 Winter Olympics?
   b. Are the data in the table categorical or quantitative? Explain your reasoning.
   c. Are the data in the table discrete or continuous? Explain your reasoning.
One way to describe a set of quantitative data is by drawing a graphical display of the data.

A **dot plot** is a data display that shows discrete data on a number line with dots, Xs, or other symbols. Dot plots help organize and display a small number of data points.

---

**WORKED EXAMPLE**

This dot plot shows the gold-medal data. The number line represents the number of gold medals. Each X above a number represents the number of countries that won that many gold medals.

![Dot Plot of 2014 Winter Olympics Gold Medal Wins by Medal-Winning Countries](image)

3. Use the dot plot to answer each question.

   a. What do the two Xs above the number 8 represent?

   b. What do the five Xs above the number 0 represent?

   c. Why are there no Xs above the number 7?

   d. Use the dot plot to determine the number of countries that won medals in the 2014 Winter Olympics. Explain your strategy.
Let’s create a dot plot to display the total number of medals won as listed in the 2014 Winter Olympics data table.

4. Make a plan for creating your dot plot.
   a. What will you name your dot plot?

   b. What numbers will begin and end your number line? Why did you select these numbers?

   c. What interval will you use on your number line? Why did you select this interval?

5. Create your dot plot displaying the data for the total medals won at the 2014 Winter Olympics.

6. Write a brief summary to report the results of your data analysis back to Maurice and Jessica to help answer their questions about gold medals and all medals won at the 2014 Winter Olympics.
These questions are part of analyzing data.

When you analyze a graphical representation of numeric data, you can look at its shape, center, and spread to draw conclusions.

- What is the overall shape of the graph? Does it have any interesting patterns?

- Where is the approximate middle, or center, of the graph?

- What does the graph tell me about how spread out the data values are?

The overall shape of a graph is called the *distribution* of data. A **distribution** is the way in which the data are spread out.

The shape of the distribution can reveal a lot of information about data. There are many different distributions, but the most common are **symmetric**, **skewed right**, and **skewed left**.
A peak is usually the value with the greatest frequency, or one of the values with the greatest frequency, and is often surrounded by data values with other large numbers of data points.

Shapes of Typical Distributions of Graphical Displays of Data

**symmetric**
- The left and right halves of the graph are mirror images of each other.
- The peak is in the middle, because there are many data values in the center.

**skewed right**
- The peak of the data is to the left side of the graph.
- There are only a few data points to the right side of the graph.

**skewed left**
- The peak of the data is to the right side of the graph.
- There are only a few data points to the left side of the graph.

1. Miko says that the dot plot shown in the previous activity for the number of gold medals won is skewed right. Do you agree with her statement? Explain your reasoning.
Examine the dot plot you created for the total number of medals won by medal-winning countries.

2. What is the distribution of the dot plot? Explain what this means in terms of the total number of medals won.

When analyzing a graphical display of data, you can also look for any interesting patterns. Some of these patterns include:

- **clusters**—areas where data are grouped close together
- **gaps**—areas where there are no data
- **peaks**—values that contain more data points than the values on either side of it
- **outliers**—data values that lie a large distance from the other data. Outliers usually accompany gaps in data.

Examine the dot plot you analyzed for the number of gold medals won by medal-winning countries.

3. Identify any clusters, gaps, peaks, or outliers. Explain what this means in terms of the number of gold medals won.

Gaps usually span multiple possible data values.
Examine the dot plot you created for the total number of medals won by medal-winning countries.

4. Identify any clusters, gaps, peaks, or outliers. Explain what this means in terms of the total number of medals won.

Another common shape for a data distribution is a uniform distribution.

Refer back to the dot plots from the Rock-Climbing Competition activity at the beginning of the lesson.

5. Describe the shape of each dot plot including its overall shape and any relevant patterns.
At the 2014 Winter Olympics, 88 countries competed in the events, but only 26 won medals. By contrast, the 2016 Summer Olympics in Rio de Janeiro, Brazil, had 207 countries compete in the events. Athletes from 80 countries won medals, but only 44 won at least 5 medals.

The table at the end of the lesson lists the total number of medals won by the top-performing countries in the 2016 Summer Olympics.

1. What comparisons can you make between the number of medals won at the 2014 Winter Olympics and the number of medals won at the 2016 Summer Olympics?

2. Do you think using a dot plot would be a good way to organize and analyze the data in the Summer Olympics table? Explain your reasoning.
A numerical data display that can easily display data sets with a larger range of data values would be helpful to plot the 2016 Summer Olympic data. A **stem-and-leaf plot** is a graphical method used to represent ordered numerical data. Once the data is ordered, the stems and leaves are determined. Typically, the stem is all the digits in a number except the rightmost digit, which is the leaf.

A stem-and-leaf plot displaying the number of medals won in the 2016 Summer Olympics is shown.

**3. Use the stem-and-leaf plot to answer each question.**

**a. Describe what you notice about the stem-and-leaf plot.**

```
<table>
<thead>
<tr>
<th>Total Medals Won by Countries</th>
<th>2016 Summer Olympics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 5 5 5 6 6 6 7 7 8 8 8 8 8 9 9</td>
<td></td>
</tr>
<tr>
<td>1 0 0 0 1 1 1 1 1 3 3 5 5 7 8 8 8 8 9 9</td>
<td></td>
</tr>
<tr>
<td>2 2 8 9</td>
<td></td>
</tr>
<tr>
<td>3 4 1 2 2</td>
<td></td>
</tr>
<tr>
<td>5 6</td>
<td></td>
</tr>
<tr>
<td>6 7</td>
<td></td>
</tr>
<tr>
<td>7 0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12 1</td>
<td></td>
</tr>
</tbody>
</table>
```

Key: 4|1 = 41 medals won.

**b. What does 7 | 0 mean in the stem-and-leaf plot?**

**c. What does 0 | 5 mean?**

**d. How many stems are in the stem-and-leaf plot?**
4. Analyze the stems and leaves in the stem-and-leaf plot.
   a. How many leaves are in the stem-and-leaf plot? Why are there that many leaves?
   b. Why would a stem have more than one leaf?
   c. Why do some stems have no leaves?
   d. Why do some stems have the same leaf repeated?
   e. Carlos claims that he should write 0s as leaves after the stems 3, 8, 9, 10, and 11 to show that there are no countries that have an amount of medals in the 30s, 80s, 90s, 100s, or 110s. Is Carlos correct? Explain your reasoning.

5. What is the most common number of medals won? How can you determine this from the stem-and-leaf plot?

6. Describe the distribution and any interesting patterns you notice in the stem-and-leaf plot. Interpret your findings in terms of the number of medals won in the 2016 Summer Olympics.

To see the distribution better, rotate the stem-and-leaf plot so that the stems resemble a horizontal number line.
During the 2016 presidential election, media reports sometimes called attention to the ages of the candidates. This led to Alicia wondering, “Are these candidates too old to be president?” Because she wanted to collect and analyze data, she revised her question: “At what age do presidents take office?”

1. Explain why Alicia’s question is a statistical question.

To answer her question, Alicia collected data on the ages when the 43 former presidents of the United States were first inaugurated. Her data is presented in the table at the end of the lesson.

To analyze the data, let’s create a stem-and-leaf plot of the former presidents’ ages at inauguration.

2. Make a plan for creating the stem-and-leaf plot.

   a. What will you choose for your stems? Why did you choose those numbers?

   b. How many leaves will you have in your stem-and-leaf plot? Explain your reasoning.

   c. Create a key for your stem-and-leaf plot. Why is this needed?
3. Create a stem-and-leaf plot to display the age at which each president was inaugurated.

4. Describe the distribution of the ages of presidents at their inaugurations.

5. The minimum age to become president of the United States is 35 years old. How is this requirement reflected in your stem-and-leaf plot?

6. What was the most common age for presidents to be inaugurated? Explain using your stem-and-leaf plot.
The four primary candidates leading into the 2016 presidential election and their projected ages at inauguration are provided.

7. How would your stem-and-leaf plot change if each person had been elected?

<table>
<thead>
<tr>
<th>Candidate</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gary Johnson</td>
<td>64</td>
</tr>
<tr>
<td>Jill Stein</td>
<td>66</td>
</tr>
<tr>
<td>Hillary Clinton</td>
<td>69</td>
</tr>
<tr>
<td>Donald Trump</td>
<td>70</td>
</tr>
</tbody>
</table>

8. Write a brief summary to report the results of your data analysis back to Alicia in response to her question about the ages of presidents at their inaugurations.
Peaks, Gaps, and Clusters... Oh, My!

Consider the six data displays shown.

1. Which plot or plots illustrate each graphical feature?
   a. cluster(s)   b. gap(s)   c. outlier(s)   d. skewness

2. Select a symmetric distribution and explain how you can make it skewed left or skewed right.
Use with Activity 2.1, Creating and Analyzing Dot Plots

<table>
<thead>
<tr>
<th>County</th>
<th>Gold Medals</th>
<th>Total Medals</th>
<th>Country</th>
<th>Gold Medals</th>
<th>Total Medals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Russian Federation</td>
<td>13</td>
<td>33</td>
<td>Slovenia</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>United States</td>
<td>9</td>
<td>28</td>
<td>Japan</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>Norway</td>
<td>11</td>
<td>26</td>
<td>Italy</td>
<td>0</td>
<td>8</td>
</tr>
<tr>
<td>Canada</td>
<td>10</td>
<td>25</td>
<td>Belarus</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Netherlands</td>
<td>8</td>
<td>24</td>
<td>Poland</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>Germany</td>
<td>8</td>
<td>19</td>
<td>Finland</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Austria</td>
<td>4</td>
<td>17</td>
<td>Great Britain</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>France</td>
<td>4</td>
<td>15</td>
<td>Latvia</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Sweden</td>
<td>2</td>
<td>15</td>
<td>Australia</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Switzerland</td>
<td>6</td>
<td>11</td>
<td>Ukraine</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>China</td>
<td>3</td>
<td>9</td>
<td>Slovakia</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>South Korea</td>
<td>3</td>
<td>8</td>
<td>Croatia</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Czech Republic</td>
<td>2</td>
<td>8</td>
<td>Kazakhstan</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Use with Activity 2.3, Stem-and-Leaf Plots

<table>
<thead>
<tr>
<th>Country</th>
<th>Medals Won</th>
<th>Country</th>
<th>Medals Won</th>
</tr>
</thead>
<tbody>
<tr>
<td>United States</td>
<td>121</td>
<td>Ukraine</td>
<td>11</td>
</tr>
<tr>
<td>China</td>
<td>70</td>
<td>Poland</td>
<td>11</td>
</tr>
<tr>
<td>Great Britain</td>
<td>67</td>
<td>Sweden</td>
<td>11</td>
</tr>
<tr>
<td>Russian Federation</td>
<td>56</td>
<td>Croatia</td>
<td>10</td>
</tr>
<tr>
<td>Germany</td>
<td>42</td>
<td>Czech Republic</td>
<td>10</td>
</tr>
<tr>
<td>France</td>
<td>42</td>
<td>South Africa</td>
<td>10</td>
</tr>
<tr>
<td>Japan</td>
<td>41</td>
<td>Cuba</td>
<td>9</td>
</tr>
<tr>
<td>Australia</td>
<td>29</td>
<td>Belarus</td>
<td>9</td>
</tr>
<tr>
<td>Italy</td>
<td>28</td>
<td>Columbia</td>
<td>8</td>
</tr>
<tr>
<td>Canada</td>
<td>22</td>
<td>Iran</td>
<td>8</td>
</tr>
<tr>
<td>Netherlands</td>
<td>19</td>
<td>Ethiopia</td>
<td>8</td>
</tr>
<tr>
<td>Brazil</td>
<td>19</td>
<td>Serbia</td>
<td>8</td>
</tr>
<tr>
<td>South Korea</td>
<td>18</td>
<td>Turkey</td>
<td>8</td>
</tr>
<tr>
<td>New Zealand</td>
<td>18</td>
<td>Georgia</td>
<td>7</td>
</tr>
<tr>
<td>Azerbaijan</td>
<td>18</td>
<td>North Korea</td>
<td>7</td>
</tr>
<tr>
<td>Kazakhstan</td>
<td>17</td>
<td>Switzerland</td>
<td>7</td>
</tr>
<tr>
<td>Hungary</td>
<td>15</td>
<td>Belgium</td>
<td>6</td>
</tr>
<tr>
<td>Denmark</td>
<td>15</td>
<td>Thailand</td>
<td>6</td>
</tr>
<tr>
<td>Kenya</td>
<td>13</td>
<td>Greece</td>
<td>6</td>
</tr>
<tr>
<td>Uzbekistan</td>
<td>13</td>
<td>Romania</td>
<td>5</td>
</tr>
<tr>
<td>Spain</td>
<td>11</td>
<td>Malaysia</td>
<td>5</td>
</tr>
<tr>
<td>Jamaica</td>
<td>11</td>
<td>Mexico</td>
<td>5</td>
</tr>
</tbody>
</table>
Use with Activity 2.4, Creating and Analyzing Stem-and-Leaf Plots

<table>
<thead>
<tr>
<th>President</th>
<th>Age at First Inauguration</th>
<th>President</th>
<th>Age at First Inauguration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Washington</td>
<td>57</td>
<td>Harrison</td>
<td>55</td>
</tr>
<tr>
<td>Adams, J.</td>
<td>61</td>
<td>McKinley</td>
<td>54</td>
</tr>
<tr>
<td>Jefferson</td>
<td>57</td>
<td>Roosevelt, T.</td>
<td>42</td>
</tr>
<tr>
<td>Madison</td>
<td>57</td>
<td>Taft</td>
<td>51</td>
</tr>
<tr>
<td>Monroe</td>
<td>58</td>
<td>Wilson</td>
<td>56</td>
</tr>
<tr>
<td>Adams, J.Q.</td>
<td>57</td>
<td>Harding</td>
<td>55</td>
</tr>
<tr>
<td>Jackson</td>
<td>61</td>
<td>Coolidge</td>
<td>51</td>
</tr>
<tr>
<td>Van Buren</td>
<td>54</td>
<td>Hoover</td>
<td>54</td>
</tr>
<tr>
<td>Harrison</td>
<td>68</td>
<td>Roosevelt, F.D.</td>
<td>51</td>
</tr>
<tr>
<td>Tyler</td>
<td>51</td>
<td>Truman</td>
<td>60</td>
</tr>
<tr>
<td>Polk</td>
<td>49</td>
<td>Eisenhower</td>
<td>62</td>
</tr>
<tr>
<td>Taylor</td>
<td>64</td>
<td>Kennedy</td>
<td>43</td>
</tr>
<tr>
<td>Fillmore</td>
<td>50</td>
<td>Johnson, L.B.</td>
<td>55</td>
</tr>
<tr>
<td>Pierce</td>
<td>48</td>
<td>Nixon</td>
<td>56</td>
</tr>
<tr>
<td>Buchanan</td>
<td>65</td>
<td>Ford</td>
<td>61</td>
</tr>
<tr>
<td>Lincoln</td>
<td>52</td>
<td>Carter</td>
<td>52</td>
</tr>
<tr>
<td>Johnson, A.</td>
<td>56</td>
<td>Reagan</td>
<td>69</td>
</tr>
<tr>
<td>Grant</td>
<td>46</td>
<td>Bush, G.H.W.</td>
<td>64</td>
</tr>
<tr>
<td>Hayes</td>
<td>54</td>
<td>Clinton</td>
<td>46</td>
</tr>
<tr>
<td>Garfield</td>
<td>49</td>
<td>Bush, G.W.</td>
<td>54</td>
</tr>
<tr>
<td>Arthur</td>
<td>50</td>
<td>Obama</td>
<td>47</td>
</tr>
<tr>
<td>Cleveland</td>
<td>47</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Use with the Assignment.

<table>
<thead>
<tr>
<th>Season</th>
<th>Wins</th>
<th>Losses</th>
<th>Season</th>
<th>Wins</th>
<th>Losses</th>
</tr>
</thead>
<tbody>
<tr>
<td>2013-2014</td>
<td>38</td>
<td>44</td>
<td>1989-1990</td>
<td>41</td>
<td>41</td>
</tr>
<tr>
<td>2012-2013</td>
<td>44</td>
<td>38</td>
<td>1988-1989</td>
<td>52</td>
<td>30</td>
</tr>
<tr>
<td>2010-2011</td>
<td>44</td>
<td>38</td>
<td>1986-1987</td>
<td>57</td>
<td>25</td>
</tr>
<tr>
<td>2009-2010</td>
<td>53</td>
<td>29</td>
<td>1985-1986</td>
<td>50</td>
<td>32</td>
</tr>
<tr>
<td>2008-2009</td>
<td>47</td>
<td>35</td>
<td>1984-1985</td>
<td>34</td>
<td>48</td>
</tr>
<tr>
<td>2007-2008</td>
<td>37</td>
<td>45</td>
<td>1983-1984</td>
<td>40</td>
<td>42</td>
</tr>
<tr>
<td>2001-2002</td>
<td>33</td>
<td>49</td>
<td>1977-1978</td>
<td>41</td>
<td>41</td>
</tr>
</tbody>
</table>
Assignment

Write
Write a definition for each of the terms in your own words.

1. dot plot
2. distribution
3. symmetric
4. skewed right
5. skewed left
6. clusters
7. gaps
8. peaks
9. outliers
10. stem-and-leaf plot

Remember
Data sets have distributions that can be described according to their shape. Dot plots are ideal for small data sets. Stem-and-leaf plots are ideal for moderately sized data sets, especially if you need to see the actual data values.

Practice
The data table at the end of the lesson shows the number of wins and losses the Atlanta Hawks have had in 48 seasons in Atlanta.

1. Create a dot plot or a stem-and-leaf plot for the number of wins by the Atlanta Hawks. Be sure to name your plot and provide a key if necessary.
2. Describe the distribution of the data. Include any specific graphical features or patterns. Explain what your answer means in terms of the number of wins by the Hawks.
3. Create a dot plot or a stem-and-leaf plot for the number of losses by the Atlanta Hawks. Be sure to name your plot and provide a key if necessary.
4. Describe the distribution of the data. Include any specific graphical features or patterns. Explain what your answer means in terms of the number of losses by the Hawks.
5. Propose a win-loss record for an upcoming season that would result in a change in the overall distribution of both plots.
**Stretch**

Another type of display used to compare two data sets is a side-by-side or back-to-back stem-and-leaf plot.

1. Describe the distribution of each data set.
2. Then, use the key and the plot to list the numerical data values in each data set.

<table>
<thead>
<tr>
<th>Data Set One</th>
<th>Data Set Two</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 6</td>
<td>4</td>
</tr>
<tr>
<td>9 3 1 5</td>
<td>9</td>
</tr>
<tr>
<td>9 8 6 5 1 1 0 6 2 7</td>
<td></td>
</tr>
<tr>
<td>7 3 2 7</td>
<td>0 0 3 6 6 8 9</td>
</tr>
<tr>
<td>5 3 8</td>
<td>0 1 1 2</td>
</tr>
<tr>
<td>2 1 9</td>
<td></td>
</tr>
</tbody>
</table>

Key: 1|5|9 = 5.1 and 5.9

---

**Review**

1. Write a statistical question about each situation.
   a. vacation destinations
   b. books

2. Plot and identify 4 points on a coordinate plane that are vertices of a parallelogram. Include points in more than one quadrant. Draw the parallelogram. Write absolute value statements for the length of the base and height of your parallelogram. Then, determine the area of the parallelogram.

3. Use long division to determine each quotient.
   a. 247 ÷ 8
   b. 894 ÷ 12
WARM UP
Use the bar graph to answer each question.

1. Who found the most sharks’ teeth? How many did that person find?
2. How many total sharks’ teeth did the friends find?

You have used dot plots and stem-and-leaf plots, which are good for small data sets. How can you display data sets with a larger number of observations?
State Parks

There are over 6000 state parks in the United States. The table shows how many state parks there are in each of the states listed.

<table>
<thead>
<tr>
<th>State</th>
<th>Number of Parks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Colorado</td>
<td>42</td>
</tr>
<tr>
<td>Arizona</td>
<td>30</td>
</tr>
<tr>
<td>Nevada</td>
<td>24</td>
</tr>
<tr>
<td>Georgia</td>
<td>66</td>
</tr>
<tr>
<td>Tennessee</td>
<td>56</td>
</tr>
<tr>
<td>Alabama</td>
<td>27</td>
</tr>
<tr>
<td>Vermont</td>
<td>57</td>
</tr>
<tr>
<td>New Hampshire</td>
<td>75</td>
</tr>
<tr>
<td>Rhode Island</td>
<td>23</td>
</tr>
</tbody>
</table>

1. Create a bar graph using the data in the table.

2. Create another bar graph with the states in alphabetical order. How is this bar graph different from your previous bar graph?

3. Suppose you wanted to graph state parks according to the region of the country. How would your bar graph be different?

The remainder of this lesson is about histograms, which look similar to bar graphs.
Minneapolis and St. Paul are known as the Twin Cities because they are close to each other in Minnesota. Both cities are home to flourishing downtowns with tall buildings.

1. Look at the graph shown.

![Histogram](image)

**Number of Floors in the Tallest Buildings in the Twin Cities**

<table>
<thead>
<tr>
<th>Number of Buildings</th>
<th>Number of Floors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

a. How is this graph different from the bar graphs you have used previously?

b. What information does the histogram display? Describe the data represented in the histogram shown. Look at the title and the labels on the axes.

c. Are the data represented in the histogram discrete or continuous? Explain your reasoning.

2. Describe the distribution of the data in terms of the overall shape and the existence of peaks, clusters, and gaps.
The first vertical bar in the histogram represents 8 buildings that have at least 10 floors but fewer than 20 floors.

3. Let’s think about how the bars are displayed in the histogram.
   a. How many bins are shown?
   b. Are all the bins the same size?
   c. What does the height of each bar represent?

4. Describe the range of floors included in each of the remaining bins shown on the horizontal axis.
   - 2nd bin: interval 20–30
   - 3rd bin: interval 30–40
   - 4th bin: interval 40–50
   - 5th bin: interval 50–60

5. If a new building was constructed that had 20 floors, which bin would change? How would it change?
6. Bella says, “There are 5 buildings represented in the histogram since there are 5 bars.” Do you agree or disagree with Bella’s statement? If you do not agree with Bella, estimate how many buildings are represented in the histogram.

7. Can you determine how many buildings have 31 floors? Explain your reasoning.

8. Is it possible to determine the number of buildings that have more than 35 floors from the histogram? Why or why not?

9. Is it possible to determine the range of the data set from the histogram? Why or why not?
To create a histogram, data is usually organized into a *grouped frequency table*. A *grouped frequency table* is a table used to organize data according to how many times data values within a given range of values occur.

10. Complete the frequency table for the number of floors in the Twin Cities' tallest buildings.

<table>
<thead>
<tr>
<th>Floor Intervals</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10–20</td>
<td></td>
</tr>
<tr>
<td>20–30</td>
<td></td>
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<td>30–40</td>
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<td>40–50</td>
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<tr>
<td>50–60</td>
<td></td>
</tr>
</tbody>
</table>

11. Write a brief summary to report the results of your data analysis about the number of floors in the Twin Cities’ tallest buildings.

__________________________________________________________________________
__________________________________________________________________________
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__________________________________________________________________________
__________________________________________________________________________
__________________________________________________________________________
New York City has over 5800 tall buildings and is home to the fifth tallest building in the United States, the Empire State Building, which is 381 meters tall. Not to be outdone, Chicago is home to the Willis Tower, formerly known as Sears Tower. It stands an impressive 442 meters tall. So how do these big cities stack up to each other? Are there any similarities or differences in the number of floors each city’s 20 tallest buildings have?

The tables listing each city’s 20 tallest buildings are provided at the end of the lesson. Use the tables to create grouped frequency tables and histograms for each city’s tallest buildings.

1. Complete the grouped frequency tables for the number of floors in each city’s 20 tallest buildings. Then complete the histograms. Make sure that you name your tables and histograms.

<table>
<thead>
<tr>
<th>Number of Floors</th>
<th>Frequency (f)</th>
<th>Number of Floors</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Remember, if a data value lies on one of the bounds, it should go in the bin to the right of that bound.
1. Is the data in the table discrete or continuous? Explain your reasoning.

2. What is similar about the histograms? What are the differences between the two histograms?

3. Use what you know about the distributions and patterns of a graphical display to describe what the histograms say about the number of floors in each city’s 20 tallest buildings.

Each year, the Empire State Building Run-Up (ESBRU) challenges runners to race up its stairs. You surveyed runners about their times at the end of the Run-Up. The results are shown in the table.

<table>
<thead>
<tr>
<th>Amount of Time to Complete the ESBRU (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.4</td>
</tr>
<tr>
<td>10.52</td>
</tr>
<tr>
<td>15.0</td>
</tr>
</tbody>
</table>
Shania and Trinh decide to make a histogram for the data set. The intervals they each want to use for the histogram are shown.

<table>
<thead>
<tr>
<th>Trinh</th>
<th>Shania</th>
</tr>
</thead>
<tbody>
<tr>
<td>9–10</td>
<td>9–10.9</td>
</tr>
<tr>
<td>11–12</td>
<td>11–12.9</td>
</tr>
<tr>
<td>13–14</td>
<td>13–14.9</td>
</tr>
<tr>
<td>15–16</td>
<td>15–16.9</td>
</tr>
<tr>
<td>17–18</td>
<td>17–18.9</td>
</tr>
</tbody>
</table>

2. Explain why both Trinh’s and Shania’s intervals are incorrect. Use a data value from the table to explain.
3. Create a grouped frequency table and histogram for the amount of time to complete the ESBRU.

4. Create a second grouped frequency table and histogram for the amount of time to complete the ESBRU. Use a different bin width than you used in Question 3. What do you notice?

5. What conclusions can you make about the amount of time it takes to complete the Empire State Building Run-Up? Use what you know about distributions and patterns of graphical displays.
### TALK the TALK

#### Which Plot Is Best?

Throughout this topic, you have created and analyzed a variety of numerical data displays.

1. List at least one advantage and one disadvantage of using each type of plot to display numerical data.

<table>
<thead>
<tr>
<th>Type</th>
<th>Advantage (or Use)</th>
<th>Disadvantage (or Limitation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dot plot</td>
<td></td>
<td></td>
</tr>
<tr>
<td>stem-and-leaf plot</td>
<td></td>
<td></td>
</tr>
<tr>
<td>histogram</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Use with Activity 3.2, Creating and Analyzing Histograms

<table>
<thead>
<tr>
<th>New York City’s 20 Tallest Buildings</th>
<th>Chicago’s 20 Tallest Buildings</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Name of Building</strong></td>
<td><strong>Number of Floors</strong></td>
</tr>
<tr>
<td>One World Trade Center</td>
<td>104</td>
</tr>
<tr>
<td>432 Park Avenue</td>
<td>89</td>
</tr>
<tr>
<td>Empire State Building</td>
<td>103</td>
</tr>
<tr>
<td>Bank of America Tower</td>
<td>54</td>
</tr>
<tr>
<td>Three World Trade Center</td>
<td>80</td>
</tr>
<tr>
<td>Chrysler Building</td>
<td>77</td>
</tr>
<tr>
<td>The New York Times Building</td>
<td>52</td>
</tr>
<tr>
<td>One57</td>
<td>75</td>
</tr>
<tr>
<td>Four World Trade Center</td>
<td>74</td>
</tr>
<tr>
<td>70 Pine Street</td>
<td>66</td>
</tr>
<tr>
<td>30 Park Place</td>
<td>82</td>
</tr>
<tr>
<td>40 Wall Street</td>
<td>70</td>
</tr>
<tr>
<td>Citigroup Center</td>
<td>59</td>
</tr>
<tr>
<td>10 Hudson Yards</td>
<td>52</td>
</tr>
<tr>
<td>8 Spruce Street</td>
<td>76</td>
</tr>
<tr>
<td>Trump World Tower</td>
<td>72</td>
</tr>
<tr>
<td>30 Rockefeller Center</td>
<td>70</td>
</tr>
<tr>
<td>56 Leonard Street</td>
<td>57</td>
</tr>
<tr>
<td>CitySpire Center</td>
<td>75</td>
</tr>
<tr>
<td>28 Liberty Street</td>
<td>60</td>
</tr>
</tbody>
</table>
Assignment

Write
Write a definition for each term in your own words.
1. histogram
2. grouped frequency table

Remember
Histograms are used when the data is numerical. Numerical data can be represented continuously in intervals.
The intervals in a histogram must all be the same size. The width of the bar represents the interval. The height of the bar indicates the frequency of values in the interval.

Practice
Jeremy's scores for the first 20 times he played the card game, Clubs and Swords, are listed.
50, 199, 246, 356, 89, 210, 391, 325, 273, 260, 100, 172, 123, 167, 194, 172, 23, 426, 75, 239
1. Create a frequency table and a histogram to display Jeremy’s scores. Be sure to name your histogram.
2. Describe the distribution of the data. Include any specific graphical features or patterns. Explain what your answer means in terms of Jeremy’s scores.
3. Create a second frequency table and histogram to provide a different view of the data distribution.

Stretch
Aviana claims that she can turn any stem-and-leaf plot into a histogram. Is she correct? Provide an example or a counterexample.
Review

1. Describe the shape of each histogram.

a. Amusement Parks, Theme Parks, Water Parks, and Zoos in the U.S.

b. Test Scores for Mr. Watson’s Math Test

2. A free diver is diving at a constant rate of 0.75 feet per second. Write and graph an equation that represents the situation.

3. Tell a story to describe the graph.

4. Determine the absolute value of each number.
   a. $|-4.2|$
   b. $|\frac{17}{8}|$
Statistical problem solving begins with a statistical question. A **statistical question** is a question that anticipates an answer based on data that vary. **Data** are categories, numbers, or observations gathered in response to a statistical question. Statistics is a problem-solving process because it is about determining a possible answer to a question that has variability. In statistics, **variability** means that the value of the attribute being studied can change from one person or thing to another.

The **statistical process** has four components:

1. Formulating a statistical question.
   
   The statistical question posed should anticipate answers that will vary. Example: How many members do the clubs at my school have? Non-example: How many students are in the Chess Club?
2. Collecting appropriate data.

Two types of variable data that can be collected are categorical and quantitative data. **Categorical data**, or qualitative data, are data for which each piece of data fits into exactly one of several different groups or categories. **Quantitative data**, or numerical data, are data for which each piece of data can be placed on a numerical scale and compared.

A statistical question can be answered by collecting data from an entire population or, more commonly, from a sample of the population. A **population** is an entire set of items from which data are collected. A **sample** is a selection from a population.

Three common methods of data collection are surveys, observational studies, and experiments. In a **survey**, people are asked one or more questions. Similarly, in an **observational study**, the researcher collects data by observing the variable of interest. In an **experiment**, the researcher imposes a condition and observes the results.

3. Analyzing the data graphically and numerically.

After you collect the data, it is time to analyze and interpret the results. Analysis includes selecting the most appropriate graphical display and numerical summaries for your question and your method of data collection.

A **bar graph** displays categorical data using either horizontal or vertical bars on a graph. The height or length of each bar indicates the value for that category.

A **circle graph**, often called a pie chart, displays categorical data using sectors, or “wedges,” of a circle. It shows how parts of the whole relate to the whole and how parts of the whole relate to the other parts. The area of each sector corresponds to the percentage of the part in relation to the whole.
To create graphs, you can determine the frequency of each response to a statistical question and record the frequencies in a frequency table. A frequency is the number of times an item or number occurs in a data set. Once the frequency is known, you can determine the mode. The mode is the value or values that occur most frequently in a data set.

4. Interpreting the results of the analysis.

You can use your analysis to make conclusions about the data. For example, from the graphs and table above you can conclude that basketball is the most popular sport among those that were surveyed.

<table>
<thead>
<tr>
<th>Sport</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Football</td>
<td>4</td>
</tr>
<tr>
<td>Softball</td>
<td>3</td>
</tr>
<tr>
<td>Basketball</td>
<td>7</td>
</tr>
<tr>
<td>Baseball</td>
<td>2</td>
</tr>
<tr>
<td>Wrestling</td>
<td>1</td>
</tr>
<tr>
<td>Gymnastics</td>
<td>5</td>
</tr>
<tr>
<td>Volleyball</td>
<td>3</td>
</tr>
<tr>
<td>Track</td>
<td>3</td>
</tr>
<tr>
<td>Swimming</td>
<td>2</td>
</tr>
</tbody>
</table>
One way to describe a set of quantitative data is by drawing a graphical display of the data.

A **dot plot** is a data display that shows discrete data on a number line with dots, Xs, or other symbols. Dot plots help organize and display a small number of data points.

In this example of a dot plot, the number line represents the number of gold medals won by countries in the 2014 Winter Olympics. Each X above a number represents a country that won that many gold medals.

When you analyze a graphical representation of numeric data, you can look at its shape, center, and spread to draw conclusions.

The overall shape of a graph is called the distribution of data. A **distribution** is the way in which the data are spread out. The shape of the distribution can reveal a lot of information about data. There are many different distributions, but the most common are **symmetric**, **skewed right**, and **skewed left**.
When analyzing a graphical display of data, you can also look for any interesting patterns. Some of these patterns include:

- **clusters**—areas where data are grouped close together
- **gaps**—areas where there are no data
- **peaks**—values that contain more data points than the values on either side of it
- **outliers**—data values that lie a large distance from the other data. Outliers usually accompany gaps in data.

### Shapes of Typical Distribution of Graphical Displays of Data

- **symmetric**
  - The left and right halves of the graph are mirror images of each other.
  - The peak is in the middle, because there are many data values in the center.

- **skewed right**
  - The peak of the data is to the left side of the graph.
  - There are only a few data points to the right side of the graph.

- **skewed left**
  - The peak of the data is to the right side of the graph.
  - There are only a few data points to the left side of the graph.

### Total Medals Won by Countries 2016 Summer Olympics

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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<tbody>
<tr>
<td></td>
<td>5</td>
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<td>11</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>12</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Key: 4|1 = 41 medals won.

A **stem-and-leaf plot** is a graphical method used to represent ordered numerical data sets with a larger range of data values. Once the data is ordered, the stem and leaves are determined. Typically, the stem is all the digits in a number except the rightmost digit, which is the leaf.
A histogram is a graphical way to display quantitative or numerical data using vertical bars. The numerical data are represented continuously with intervals. The intervals in a histogram must all be the same size. The width of a bar in a histogram represents the interval. The height of the bar indicates the frequency, or the number of data values, in the interval.

Dot plots show individual data values. Histograms display grouped data. For example, you cannot determine from the histogram how many buildings have 21 floors, or more than 45 floors.

To create a histogram, data is usually organized into a grouped frequency table. A grouped frequency table is a table used to organize data according to how many times data values within a given range of values occur.

For example, this grouped frequency table displays the data represented by the histogram above.

<table>
<thead>
<tr>
<th>Floor Intervals</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10–20</td>
<td>8</td>
</tr>
<tr>
<td>20–30</td>
<td>27</td>
</tr>
<tr>
<td>30–40</td>
<td>16</td>
</tr>
<tr>
<td>40–50</td>
<td>5</td>
</tr>
<tr>
<td>50–60</td>
<td>4</td>
</tr>
</tbody>
</table>
TOPIC 2
Numerical Summaries of Data

The size of vertebrates varies widely, from blue whales at the large end to tiny frogs at the small end.

Lesson 1
In the Middle
Analyzing Data Using Measures of Center ........................................ M5-71

Lesson 2
Box It Up
Displaying the Five-Number Summary ........................................ M5-87

Lesson 3
March MADness
Mean Absolute Deviation ............................................................ M5-105

Lesson 4
You Chose... Wisely
Choosing Appropriate Measures ........................................ M5-117
TOPIC 2: NUMERICAL SUMMARIES OF DATA

In this topic, students learn about measures of central tendency and measures of variability and when each is the most appropriate measure for a given data set. Students may have an informal or intuitive understanding of “average,” but this topic formalizes the ideas of the mean and median of a data set. They learn that the median is the middle value in a data set and that the mean can be thought of as a fair share or the balance point of a data set. From there, students learn about measures of variability, specifically the interquartile range and mean absolute deviation. Students analyze data sets, selecting the most appropriate measures of central tendency and measures of variability.

Where have we been?
In prior grades, students determined which value in a data set occurred the most. This measure is the mode. Students build on that as they learn about other measures of central tendency: the median and the mean. When students learned about division, they created equal groups, which is a similar construct to mean as “fair share.”

Where are we going?
This topic provides students with the building blocks of numerical data analysis: calculating measures of central tendency and measures of variability to describe data. Students will continue to use these computations, and the reasoning behind them, as they compare data distributions in grade 7.

Using Models to Determine the Mean of a Data Set

The model shows how fair shares can be used to visualize the mean of a data set. For example, this simplified data set shows 2 and 6. By moving 2 from the stack of 6 to the stack of 2, the data are evenly distributed, with 4 in each stack. Thus, the mean of 2 and 6 is 4.
Myth: Some students are “right-brain” learners while other students are “left-brain” learners.

As you probably know, the brain is divided into two hemispheres: the left and the right. Some categorize people by their preferred or dominant mode of thinking. “Right-brain” thinkers are considered to be more intuitive, creative, and imaginative. “Left-brain” thinkers are more logical, verbal, and mathematical.

The brain can also be broken down into lobes. The occipital lobe can be found in back of the brain, and it is responsible for processing visual information. The temporal lobes, which sit above your ears, process language and sensory information. A band across the top of your head is the parietal lobe, and it controls movement. Finally, the frontal lobe is where planning and learning occurs. Another way to think about the brain is from the back to the front, where information goes from highly concrete to abstract.

Why don’t we claim that some people are “back of the brain” thinkers who are highly concrete; whereas, others are “frontal thinkers” who are more abstract? The reason is that the brain is a highly interconnected organ. Each lobe hands off information to be processed by other lobes, and they are constantly talking to each other. All of us are whole-brain thinkers!

#mathmythbusted

Talking Points
You can support your student’s learning by asking questions about the work they do in class or at home. Your student is learning about the process of framing questions about data and representing data numerically.

Questions to Ask
• How does this problem look like something you did in class?
• Can you show me the strategy you used to solve this problem? Do you know another way to solve it?
• Does your answer make sense? Why?
• Is there anything you don’t understand? How can you use today’s lesson to help?

Key Terms
median
The median is the middle number in a data set when the values are placed in order from least to greatest.

mean
The mean is the arithmetic average of the numbers in a data set.

range
The range is the difference between the maximum and minimum values of a data set.
In the Middle

Analyzing Data Using Measures of Center

WARM UP
Simplify each numeric expression.

1. \((13 + 17) \div 2\)
2. \((29 + 36) \div 2\)
3. \((48 + 9) \div 2\)
4. \((27 + 31) \div 2\)

LEARNING GOALS
• Define the three measures of center: mode, median, and mean.
• Recognize that a measure of center for a numerical data set is a single value that summarizes all of its values.
• Give quantitative measures of center for a data set, including mean and/or median, and interpret the mode, median, and mean for a data set.

KEY TERMS
• measure of center
• mode
• median
• balance point
• mean

You have analyzed, created, and interpreted data displays such as dot plots, stem-and-leaf plots, and histograms. You have described shapes and patterns in distributions of data displays. How can you describe a numerical data set as a single value?
Describing Data

Analyze each display. Identify the most typical value and estimate the middle value in each.

1. **Pencils in Backpacks**

```
  X  X  X  X  X  X  X  X  X  X  X  X  X  X
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
Number of Pencils
```

2. **Ages of U.S. First Ladies (20th Century)**

```
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0 3 4 5 7 8 9</td>
</tr>
<tr>
<td>5</td>
<td>0 2 4 6 6 6 9</td>
</tr>
<tr>
<td>6</td>
<td>0 0 3</td>
</tr>
</tbody>
</table>
```

Key: 6|0 means 60.

3. **Hours Spent Playing Video Games on Weekends**

```
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 3</td>
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<tr>
<td>3</td>
<td>2 3</td>
</tr>
<tr>
<td>4</td>
<td>3 4</td>
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<td>5</td>
<td>3 4</td>
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<td>6</td>
<td>3 4</td>
</tr>
<tr>
<td>7</td>
<td>3 4</td>
</tr>
<tr>
<td>8</td>
<td>3 4</td>
</tr>
<tr>
<td>9</td>
<td>3 4</td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
```

What patterns do you notice in the data?
When you analyze a set of data, you often want to describe it numerically. One way to numerically describe a data set is to use a measure of center.

There are three measures that describe how a data set is centered: the mean, the median, and the mode.

The mode is the data value or values that occur most frequently in a data set.

The median is the middle number in a data set when the values are placed in order from least to greatest or greatest to least. When a data set has an odd number of data values, you can determine which number is exactly in the middle of the data set. If there is an even number of data values, then the median is calculated by adding the two middle numbers and dividing by 2.

1. The Olive Street Middle School girls’ basketball team has a chance to be in the league playoffs. Coach Harris must determine which of the following 3 players should get more playing time in the first playoff game. In the past six games, Josephine scored 12, 12, 6, 26, 4, and 12 points. Shelly scored 3, 2, 8, 17, 10, and 20 points. Chanice scored 15, 12, 13, 10, 8, and 14 points.

a. Determine the mode for the number of points scored by each player.

b. Determine the median number of points scored by each player.
c. Which of these two measures of center, mode or median, would be better for Coach Harris to use in making her decision? Explain your reasoning.

2. Explain what Lamar did incorrectly to determine that the median was 10. Then determine the correct median.

Lamar says that the median is 10 for the data set 5, 6, 10, 4, and 9.
A Third Measure: Mean as Fair Share

There is a third measure of center that can describe the values in a data set. This measure of center is called the mean and is based on leveling off or creating fair shares.

WORKED EXAMPLE

Analyze the two stacks of cubes.

If you were to create two equal stacks of cubes, you would subtract two cubes from the greater stack, and add the two cubes to the lesser stack. In doing so, you have created two equal stacks.
1. Analyze each stack of cubes shown. Create four equal stacks of cubes. Record what operations you performed.

a. 2, 3, 5, 6

You have to keep the number of stacks the same.

b. 2, 3, 5, 10

2. Compare your results from parts (a) and (b). How did the number of cubes in each equal stack change in part (b)? Explain why this happened.
In the previous activity, data values were represented by stacks of cubes. You rearranged the stacks to create equal stacks, or fair shares. You can also represent quantities on a number line and create a balance point.

**WORKED EXAMPLE**

Consider the data set: 2, 6.

\[2 + 2 = 4\]
\[6 - 2 = 4\]

The value 2 was moved to the right from 2 to 4. To maintain balance, 6 was moved 2 to the left from 6 to 4. The balance point is 4.

When you are attempting to create a balance on a number line, if you move a value to the right a certain amount, then you must also move a value to the left that amount. You can move a data value to the left and right as much as you like as long as you do the opposite to another data value. You can start however you like.

1. **What do you think the ▲ represents?**

You can also determine the balance point of a number line with more than two data points.
2. Kathryn determined the balance point of the data set. Record the operations she used on the blank lines in each step. Label the balance point in Step 3.
Recall the data sets for the number of points each player scored for the Olive Street Middle School Basketball team.

3. For each data set, determine the balance point on the number lines shown. Record the steps you used to determine the balance point.

a. Josephine
   Data set: 12, 12, 6, 26, 4, 12

![Number line for Josephine's data set]

b. Shelly
   Data set: 3, 2, 8, 17, 10, 20

![Number line for Shelly's data set]

c. Chanice
   Data set: 15, 12, 13, 10, 8, 14

![Number line for Chanice's data set]
The balance point can also be called the mean. The mean is the arithmetic average of the numbers in a data set.

**WORKED EXAMPLE**

The mean is calculated by adding all of the values in the data set and dividing the sum by the number of values.

The mean of the data set for the points Josephine scored is calculated by:

**Step 1:** $12 + 12 + 6 + 26 + 4 + 12 = 72$

**Step 2:** $\frac{72}{6} = 12$

You can verify that the mean is 12 because the balance point of the data set is 12.

4. Calculate the mean number of points scored by each player.

a. Josephine
   Data set: 12, 12, 6, 26, 4, 12

b. Shelly
   Data set: 3, 2, 8, 17, 10, 20

b. Chanice
   Data set: 15, 12, 13, 10, 8, 14
A corporation is awarding grants to local schools to purchase fitness equipment. The principal at Sharpe Middle School would like to submit an application for the grant. If awarded the money, the school would like to add fitness equipment to the gym.

Before she submits the application, the principal wants to understand how much time the students at Sharpe Middle School spend exercising each weekday. She decides to give a survey to 15 anonymous students. The results are shown.

0 minutes  40 minutes  60 minutes  30 minutes  60 minutes
10 minutes  45 minutes  30 minutes  300 minutes  90 minutes
30 minutes  120 minutes  60 minutes  0 minutes  20 minutes

1. **Identify the statistical question posed in this situation.**
   Create a display from the survey data. Calculate and interpret each measure of center. Then write a summary statement based on your findings.
**TALK the TALK**

**Describing a Numerical Data Set as a Single Value**

In this lesson you learned about three measures of center: mode, median, and mean.

1. Describe how you can use each measure of center to describe a data set.

2. What are the most important differences between each of the measures of center?

3. Which is your favorite measure of center? Why?
Write

Choose a word to best complete each sentence.

measure of center  mode  median  mean  balance point

1. A ______________ for a numerical data set summarizes all of its values with a single number.
2. The ______________ is the arithmetic average of the numbers in a data set.
3. When you have all the points on a number line at the same value after moving data values, this value is called the ______________.
4. The ______________ is the middle number in a data set when the values are placed in order from least to greatest.
5. The ______________ is the data value or values that occur most frequently in a data set.

Remember

There are three measures of center: mode, median, and mean. Measures of center are numerical ways of determining where the center of data is located.

Practice

1. Determine the mode and median for each data set. What does each measure of center tell you about the data set?
   a. The heights of each of your classmates in inches are 62, 58, 67, 68, 68, 72, 66, 65, 60, 61, 64, 67, and 64.
   b. Yolanda made golf putts from distances of 7 feet, 15 feet, 8 feet, 9.5 feet, and 11 feet from the hole.
   c. Everyone in your class reaches into their pockets to see how much change they have. The amounts, in cents, are 15, 48, 92, 72, 65, 60, 61, 64, 67, and 64.

2. Ms. Zhang’s math class has had 5 quizzes, each worth 10 points. Julian and his friends, Mona and Timi, are determining who did the best on the quizzes. Their scores are:
   Julian: 3, 9, 9, 9, 10
   Mona: 6, 7, 7, 10, 10
   Timi: 6, 7, 8, 9, 10
   a. According to the mode, who did the best on the quizzes? Is the mode a good way to determine who did best on the quizzes? Why or why not?
   b. According to the median, who did the best on the quizzes? Is the median a good way to determine who did best on the quizzes? Why or why not?
c. Determine the mean score by leveling off. Show diagrams and record your operations.
d. Determine each student’s mean by determining the balance point. Show your steps on a number line and record your operations.
e. Calculate each student’s mean score.
f. Who would you say did best on the quizzes? Explain your choice.

3. The rate at which crickets chirp is affected by the temperature. In fact, you can estimate the outside temperature by counting cricket chirps. As a homework assignment, Mr. Ortega asks each of his students to count the number of chirps they hear in 15 seconds at 8:00 pm. The results are shown.
   36, 37, 41, 39, 35, 39, 35, 39, 42, 37, 40, 35, 36, 37, 42, 35, 37, 37, 38, 42, 41, 37, 41
a. Determine the mode for the number of chirps heard in 15 seconds.
b. What does the mode tell you about the number of chirps heard in 15 seconds?
c. Determine the median number of chirps heard in 15 seconds.
d. What does the median tell you about the number of chirps heard in 15 seconds?
e. Calculate the mean number of chirps heard by the students in 15 seconds.
f. What does the mean tell you about the number of chirps heard in 15 seconds?

4. An estimate of the temperature outside can be calculated by adding 40 to the number of cricket chirps you hear in 15 seconds. Chelsea used a sample of six calculated chirps:
   76, 74, 74, 76, 73, 77
a. Use a number line to determine the balance point to estimate the mean number of chirps. Then, describe the steps you took to determine the balance point.
b. Calculate the mean from the data values. How does it compare to your answer from part (a)?

Stretch
1. Create a data set where the mean is greater than the median.
2. Create a data set where the mean is less than the median.
Review

1. Describe the shape of the distribution of the data shown.

   a. 1|4 – 14
      
      |   |
      1  4
      2  0 1
      3  2 2 9
      4  1 4 5 6 7
      5  3 5 7 7

   b. 4|1 = 4.1
      
      |   |
      4  1
      5  3 7 8
      6  0 1 4 4
      7  1 3 4 7
      8  0 9 9
      9  6

2. The graph represents the total distance traveled in miles. Use the graph to answer each question.

   a. Write an equation to represent the graph. Define each variable.
   b. If you know the number of hours, how can you use the graph to determine any total distance?

   How many hours did it take to travel 120 miles?

3. Evaluate each expression.
   a. \( \frac{(8 + 2)^2}{2} \)
   b. \( 10 + 5^2 - 3 \cdot 2^2 \)
LEARNING GOALS

• Give quantitative measures of variation, including interquartile range, and interpret the range, quartiles, and interquartile range as measures of variation for a data set.
• Calculate and interpret the five-number summary as a measure of variation for a data set.
• Display numerical data in box plots.
• Describe an overall pattern of data with reference to the context in which the data were gathered.

KEY TERMS

• measures of variation
• range
• quartile
• interquartile range (IQR)
• box-and-whisker plot

WARM UP

The scores from an English quiz are displayed in the stem-and-leaf plot.

<table>
<thead>
<tr>
<th>Stem</th>
<th>Leaf</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>3 4 5</td>
</tr>
<tr>
<td>9</td>
<td>2 3</td>
</tr>
</tbody>
</table>

Key
5\|8 = 58%

1. What is the median of the data?
2. What does the median describe in the problem situation?

Mean, median, and mode are used to describe measures of center for a data set. Other characteristics are also important, such as how much the data varies from that center. How can you use mathematics to describe the variation in a data set?
Human Box Plot

You teacher will provide you with an index card and a penny. On the index card, write the date imprinted on the penny.

1. Consider the data from the class and predict the shape of the data set. Do you think that it will be skewed right, skewed left, or symmetrical? Do you think that there will be any clusters or gaps?

On your teacher’s signal, line up with your index card from oldest date to the most recent date. As a class, discuss how to determine the following measures.

2. Complete the table for your data set.

<table>
<thead>
<tr>
<th>Minimum</th>
<th>Lower Quartile (Q1)</th>
<th>Median</th>
<th>Upper Quartile (Q3)</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Use what you learned in this activity to write descriptions of each term.

a. minimum

b. lower quartile (Q1)

c. median

d. upper quartile (Q3)

e. maximum
Given the collection of pennies provided by your teacher, line up the pennies from the earliest imprinted date to the most recent.

1. Describe the variation of the data.

A measure of variation describes the spread of data values. One measure of variation is the range. The range is the difference between the maximum and minimum values of a data set.

2. Identify the minimum and maximum values of the data set.

3. Calculate the range for the penny data.
Another set of values that helps to describe variation in a data set is a **quartile**. When data in a set is arranged in order, **quartiles** are the numbers that split data into quarters (or fourths).

Quartiles are often denoted by the letter Q followed by a number that indicates which fourth it represents. Since the median is the second quartile, it could be denoted Q2. The other quartiles are Q1 and Q3.

4. How do you calculate the quartiles for a data set?

5. How many quartiles does it take to divide the data into fourths?

In your workspace, divide your pennies into quartiles, leaving a space for each.

6. Identify the value of each quartile for the penny data in your group.
   a. Q1
   b. Q2
   c. Q3
7. For each quartile, identify what percent of the data is below the quartile and what percent of the data is above the quartile.

    below            above

    a. Q1

    b. Q2

    c. Q3

8. What percent of the data is between Q1 and Q3?

The interquartile range, abbreviated IQR, is the difference between the third quartile, Q3, and the first quartile, Q1. The IQR indicates the range of the middle 50 percent of the data.

9. What do you think is meant by the middle 50 percent?

10. Do you think it is possible for two sets of data to have the same range, but different IQRs? Explain your reasoning.
11. What is the IQR for your penny data?

To summarize and describe the spread of the data values, you can use the five-number summary. The five-number summary includes these 5 values from a data set:

- Minimum: the least value in the data set
- Q1: the first quartile
- Median: the median of the data set
- Q3: the third quartile
- Maximum: the greatest value in a data set.

12. Determine the 5-number summary and IQR for each data set. Explain the process you used to calculate the values and what they tell you about the data.

a. 24, 32, 16, 18, 30, 20

b. 200, 150, 260, 180, 300, 240, 280
There is a special type of graph that displays the variation in a data set. A **box-and-whisker plot**, or just box plot, is a graph that displays the five-number summary of a data set.

Examine the box-and-whisker plot shown.

**WORKED EXAMPLE**

Recall that the five-number summary consists of:
- minimum value in the data set
- Q1
- median
- Q3
- maximum value in the data set

1. What does the “box” in the box-and-whisker plot represent? What do the “whiskers” represent?
Box-and-whisker plots can be represented horizontally and vertically.

2. Use the box-and-whisker plot shown to answer each question.

![Box-and-whisker plot]

Number of Points Scored on a Math Test

a. Identify the given values for the points scored on the math test. Then, explain what those values tell you about the scores on the test.

- minimum: • Q1:
- median: • Q3:
- maximum: • range:

b. Determine the IQR for the test scores. Then, explain what the IQR represents in this problem situation.

c. How many students took the math test? Explain your answer.
d. Karyn says the median should be at 50 because it is in the middle of the number line. Do you agree with Karyn's claim? Explain how you determined your answer.

e. Jamal claims that more students scored between 15 and 40 than between 70 and 90 because the lower whisker is longer than the upper whisker. Do you agree or disagree with Jamal? Explain your reasoning.

You can describe the distribution of a box plot in the same way you described the shapes of stem-and-leaf plots or histograms.

3. How would you describe the distribution of the box plot? Why?

4. Create a box-and-whisker plot to represent the penny data. First, recall the values of the five-number summary.

a. minimum:  

b. Q1:

c. median:  

d. Q3:

e. maximum:
5. Use the number line shown to complete the box-and-whisker plot.

![Number Line Diagram](image_url)

a. Describe the distribution of the box plot.

b. How do the lengths of the whiskers compare? Describe why you think this is so.

6. Think about the line of pennies that you created when you ordered them by imprinted dates and separated them into quartiles.

a. How many pennies were below Q1? How many pennies were above Q3?

b. Do these values match the length of the whiskers on your box-and-whisker plot?

7. Summarize what you can determine about the penny data by examining the box plot.
A newspaper reporter is writing an investigative story about the wait time at two local restaurants. With the help of her assistant, the reporter randomly selected 11 patrons at each restaurant and recorded how many minutes they had to wait before being served. The results are shown.

1. Create a vertical box-and-whisker plot for the wait times at each of the restaurants. Use the same number line for each representation so that they can be compared.

2. Describe the distributions of each box plot.

3. What is the range of wait times? What does the five-number summary tell you about the spread of the data that the range does not tell you?
4. What do the IQR values tell you about the time spent waiting at each restaurant?

5. How does the mean wait time compare to the median wait time for each restaurant?

6. Assume the food prices and service were the same in both restaurants. Write a brief summary to report the results of your data analysis back to the newspaper reporter to help answer their question about wait times in The Captain’s Corner and The First Deck restaurants.

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

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________________________________________________________________________

________________________________________________________________________
TALK the TALK

Build a Box

1. Analyze the box-and-whisker plot shown. Determine if each statement is true or false and provide a reason for your decision.

- a. On average, the girls are taller.
- b. The range of heights is greater for the boys.
- c. Half the girls are over 165 cm tall.
- d. Half the boys are over 172 cm tall.
- e. The shortest person is a boy.
- f. The tallest person is a boy.
2. Use the given information to determine a five-number summary and construct a box-and-whisker plot. Is your data set the only possible solution? Why or why not?

- The data set has a range of 30.
- The maximum value is 50.
- The IQR is 10.
- The median is closer to Q1 than to Q3.

3. Create a data set to represent the box-and-whisker plot shown. Is your data set the only possible solution? Why or why not?

![Box-and-whisker plot]

a. Data set with 11 numbers.

b. Data set with 8 numbers.
**Assignment**

**Write**
Write the term that best completes each statement.

<table>
<thead>
<tr>
<th>range</th>
<th>quartiles</th>
<th>interquartile range</th>
<th>five-number summary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The ____________ is the difference between the first quartile and the third quartile.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. The ____________ for a set of data is the difference between the maximum and minimum values.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. ____________ are values that divide a data set into four equal parts once the data are arranged in ascending order.</td>
<td></td>
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</tr>
<tr>
<td>4. A(n) ____________ lists the minimum and maximum values, the median, and the quartiles for a set of data.</td>
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</tbody>
</table>

**Remember**
To summarize and describe the spread of data values, you can use a five-number summary. A five-number summary includes 5 values from a data set:

- Minimum: the least value in the data set
- Q1: the first quartile, or the median of the lower half of data
- Median: the median of the data set
- Q3: the third quartile, or the median of the upper half of data
- Maximum: the greatest value in a data set.

The representation of a five-number summary is called a box-and-whisker plot, or simply a box plot.

**Practice**
1. The box-and-whisker plot shows the distribution of scores on a history quiz.

   ![Box-and-Whisker Plot](image)

   a. Identify the median of the data and interpret its meaning.
   b. Identify the range of the data and interpret its meaning.
2. Answer each question using the data set: 0, 5, 5, 15, 30, 30, 45, 50, 50, 60, 75, 110, 140, 240, 330.
   a. Sketch a box-and-whisker plot.
   b. What is the median for the data set?
   c. What is Q3 for the data set?
3. Answer each question using the data set: 10, 10, 10, 10, 35, 90, 95, 100, 175, 420, 490, 515, 515, 790.
   a. Sketch a box-and-whisker plot.
   b. What do you notice about this box-and-whisker plot?
   c. What is the median for the data set?
4. The residents of Summersville, West Virginia, are concerned about people speeding through their town on US Route 19. The police decide to monitor the speed of the cars that pass through the town at various times during the day. The data show the recorded speeds in miles per hour of 23 cars at 7:30 am on one Wednesday morning.
   73, 68, 72, 61, 51, 68, 70, 53, 72, 71, 46, 51, 55, 53, 65, 57, 65, 57, 58, 68, 61, 48, 83
   a. What is the range of data?
   b. Construct a box-and-whisker plot of the data.
   c. Interpret each number in the five-number summary.
   d. What does the IQR value tell you about the speeds of the cars?
   e. If the speed limit through the town is 50 miles per hour, should the residents be concerned based on this data?
5. San Francisco, California, and Richmond, Virginia, are located at about the same latitude on opposite sides of the United States. The table shows the amount of rainfall each city gets on average each month.

<table>
<thead>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>San Francisco, CA</strong></td>
<td>4.4 in.</td>
<td>3.3 in.</td>
<td>3.1 in.</td>
<td>1.4 in.</td>
<td>0.3 in.</td>
<td>0.1 in.</td>
<td>0.0 in.</td>
<td>0.1 in.</td>
<td>0.3 in.</td>
<td>1.3 in.</td>
<td>2.9 in.</td>
<td>3.1 in.</td>
</tr>
<tr>
<td><strong>Richmond, VA</strong></td>
<td>3.3 in.</td>
<td>3.3 in.</td>
<td>3.6 in.</td>
<td>3.0 in.</td>
<td>3.8 in.</td>
<td>3.6 in.</td>
<td>5.0 in.</td>
<td>4.4 in.</td>
<td>3.3 in.</td>
<td>3.5 in.</td>
<td>3.3 in.</td>
<td>3.3 in.</td>
</tr>
</tbody>
</table>

   a. Calculate and interpret the range of rainfall for each city.
   b. Construct box-and-whisker plots for the average rainfall for each city on the same graph.
   c. Compare the rainfall in San Francisco and Richmond. Describe the shape of each box plot.
   d. Interpret the IQR for the rainfall in each city.

**Stretch**

An arithmetic sequence is formed by adding (or subtracting) the same number over and over.

For example, 2, 5, 8, 11, 14, 17 . . . is an arithmetic sequence formed by adding 3 over and over after choosing a starting number.

1. Create box-and-whisker plots using different arithmetic sequences as data.
2. How are the plots similar?
3. What do you notice about the IQR of each plot?
Review

1. Determine the mean, median, mode, and range of the set of data. Round your answers to the nearest hundredth when necessary.
   a. 14, 19, 8, 22, 11, 19, 4, 18, 12, 10, 21
   b. 55, 24, 73, 108, 39, 46, 72, 100, 92, 32

2. Construct a dot plot for each.
   a. Display the number of items purchased by a number of randomly chosen customers at a toy store. The data are 2, 4, 3, 12, 3, 1, 5, 6, 3, 4, 2, 4, 3, 7, 14, 10, 3, 5, and 9. Describe the distribution.
   b. Display the scores on a recent math quiz. The data are 12, 14, 8, 13, 12, 14, 5, 13, 14, 3, 15, 15, 10, 13, 12, 0, 14, 11, 14, 13, and 10. Describe the distribution.

3. Determine each product.
   a. $\frac{3}{8} \times \frac{4}{5}$
   b. $\frac{1}{2} \times \frac{2}{1}$
March MADness
Mean Absolute Deviation

WARM UP
Determine the absolute value of each number.
1. |−4|
2. |12.5|
3. |−1.09|
4. |4\frac{2}{3}|

LEARNING GOALS
• Determine the absolute deviations of data points in a data set.
• Give quantitative measures of variation, including mean absolute deviation, for a data set.
• Use the mean absolute deviation as a measure of variation to describe and interpret data.
• Compare data sets using variation and the mean absolute deviation.
• Summarize numerical data sets in relation to their context.

KEY TERMS
• deviation
• absolute deviation
• mean absolute deviation

The interquartile range is used as a measure of variation when the median is the measure of center. How can you measure the variation when mean is the measure of center?
We Are the Champions

Coach Harris’s basketball team is advancing to the district championship. Tamika and Lynn are possible starters for the game. Dot plots for each player’s scoring over the past six games are shown.

1. Determine the mean of each data set. Explain what this number tells you.

2. How are the two data sets similar and different?

3. Explain why the two data sets have the same mean.
Previously, you examined the dot plots of two basketball players—Tamika and Lynn. Coach Harris needs to choose between Tamika and Lynn as starters for the game.

**Number of Points Scored by Tamika**

![Dot plot of Tamika's points scored]

- Mean = 12

**Number of Points Scored by Lynn**

![Dot plot of Lynn's points scored]

- Mean = 12

1. Based on the dot plots, which player do you think Coach Harris should choose?

When analyzing a data set, measures of center give you an idea of where the data is centered, or what a typical data value might be. There is another measure that can help you analyze data. Measures of variation describe the spread of the data values. Just as there are several measures of central tendency, there are also several measures of variation.

The deviation of a data value indicates how far that data value is from the mean. To calculate the deviation, subtract the mean from the data value:

\[
\text{deviation} = \text{data value} - \text{mean}
\]
2. Describe the deviations. Record your results in the tables.

<table>
<thead>
<tr>
<th>Tamika</th>
<th>Describe the Deviation from the Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points Scored</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
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<tr>
<td>26</td>
<td></td>
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<td>6</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lynn</th>
<th>Describe the Deviation from the Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points Scored</td>
<td></td>
</tr>
<tr>
<td>15</td>
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<td>12</td>
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<tr>
<td>13</td>
<td></td>
</tr>
</tbody>
</table>

3. What is the meaning if a deviation is positive? Is negative? Is 0?

4. What do you notice about the deviations for each player?
5. Carly claims that the sum of the deviations for a data set will always be 0. Do you agree? Why or why not?

The sum of all the deviations less than 0 is equal to the sum of the deviations greater than 0. Because the mean is the balance point, the sums of data points on either side of the balance point are equal to each other.

In order to get an idea of the spread of the data values, you can take the absolute value of each deviation and then determine the mean of those absolute values. The absolute value of each deviation is called the **absolute deviation**. The **mean absolute deviation** (MAD) is the mean of the absolute deviations.

6. Record the absolute deviations for the points scored in the tables.

<table>
<thead>
<tr>
<th>Points Scored</th>
<th>Deviation from the Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tamika</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-6</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-6</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Points Scored</th>
<th>Deviation from the Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lynn</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
7. Calculate the mean absolute deviation for the points scored for each player.

8. What does the mean absolute deviation tell you about the points scored by each player?

9. If you were Coach Harris, which player would you choose to play in the championship game? Justify your decision.

ACTIVITY 3.2 Variation in Non-Numeric Data

Sometimes you can change non-numerical data into numeric data in order to analyze it. Consider, for example, the report cards shown. Grades for the courses are assigned to the categories A, B, C, D, and F, with A being the highest grade.

<table>
<thead>
<tr>
<th></th>
<th>Luca</th>
<th></th>
<th>Eric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Science</td>
<td>B</td>
<td>Math</td>
<td>A</td>
</tr>
<tr>
<td>Cultural Literacy</td>
<td>A</td>
<td>English</td>
<td>B</td>
</tr>
<tr>
<td>Music</td>
<td>C</td>
<td>Cultural Literacy</td>
<td>C</td>
</tr>
<tr>
<td>Math</td>
<td>A</td>
<td>Science</td>
<td>A</td>
</tr>
<tr>
<td>English</td>
<td>B</td>
<td>Music</td>
<td>A</td>
</tr>
</tbody>
</table>
1. Explain how you can change the report card data into numeric data.

2. Determine the mean of each data set. What does each mean tell you?

3. Determine the mean absolute deviation for each data set.

<table>
<thead>
<tr>
<th></th>
<th>Luca</th>
<th>Eric</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>Value</td>
<td>Describe the Deviation from the Mean</td>
</tr>
<tr>
<td>Value</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
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</tr>
</tbody>
</table>

4. Interpret each of the mean absolute deviations.
The tables on the next page show the heights in inches of ten NBA basketball players and ten 6th-grade basketball players.

1. Write a statistical question you can answer by analyzing the data.

2. Create a dot plot for each data set.
3. Complete each table. Then, compare the data sets and interpret your results.

### NBA Players

<table>
<thead>
<tr>
<th>Height (in.)</th>
<th>Describe the Deviation from the Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>79</td>
<td></td>
<td></td>
</tr>
<tr>
<td>74</td>
<td></td>
<td></td>
</tr>
<tr>
<td>78</td>
<td></td>
<td></td>
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<tr>
<td>81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>81</td>
<td></td>
<td></td>
</tr>
<tr>
<td>76</td>
<td></td>
<td></td>
</tr>
<tr>
<td>84</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>82</td>
<td></td>
<td></td>
</tr>
<tr>
<td>83</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 6th-Grade Players

<table>
<thead>
<tr>
<th>Height (in.)</th>
<th>Describe the Deviation from the Mean</th>
<th>Absolute Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>68</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
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<tr>
<td>58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>62</td>
<td></td>
<td></td>
</tr>
<tr>
<td>65</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>65</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mean absolute deviation and interquartile range are both measures of variation.
TALK the TALK

IQR and MAD

1. Create vertical box plots and calculate the interquartile ranges for the data sets in the previous activity.

2. Compare the mean absolute deviations and the interquartile ranges.
   a. What does each measure tell you about the data set?
   b. How are they the same? How are they different?
Assignment

Write
Complete each sentence with the correct term.

<table>
<thead>
<tr>
<th>Absolute deviation</th>
<th>Mean absolute deviation</th>
<th>Measures of variation</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ___________ describe(s) the spread of the data values.</td>
<td>2. ___________ indicates how far the data value is from the mean.</td>
<td>3. ___________ is the absolute value of each deviation.</td>
<td>4. ___________ is the average, or mean, of the absolute deviations.</td>
</tr>
</tbody>
</table>

Remember
To calculate the mean absolute deviation:

- Determine the mean of the data.
- Subtract the mean from each data value. These are the deviations.
- Record the absolute value of each deviation. These are the absolute deviations.
- Determine the mean of the absolute deviations. This is the mean absolute deviation.

Practice
Calculate the mean absolute deviation for each data set.
1. Data set: 4, 5, 9, 4, 8; Mean = 6
2. Data set: 7, 11, 8, 35, 14; Mean = 15
3. Data set: 60, 65, 66, 67, 67, 65; Mean = 65
4. Data set: 22, 26, 29, 21, 28, 24, 25, 26; Mean = 25
5. Data set: 180, 210, 155, 110, 230, 90, 400, 35, 190, 0, 10, 100, 90, 130, 200; Mean = 142
6. Data set: 55, 74, 90, 20, 47, 59, 26, 83, 77, 62, 58, 33, 57, 44, 31; Mean = 54.4

Stretch
1. Create a data set of 5 numbers that has a mean absolute deviation of 1. Explain how you arrived at your solution.
2. Create a data set of 6 numbers that has a mean absolute deviation of 10. Explain how you arrived at your solution.
Review
1. The rate at which crickets chirp is affected by the temperature. In fact, you can estimate the outside temperature by counting cricket chirps. As a homework assignment, Mr. Ortega asks each of his students to count the number of chirps they hear in 15 seconds at 8:00 pm. The results are shown.

36, 37, 41, 39, 35, 39, 39, 35, 36, 42, 37, 40, 35, 37, 42, 35, 37, 38, 42, 37, 41

Determine the median and mean number of cricket chirps heard in 15 seconds.

2. Patrick recorded the number of emails he sent over two weeks: 11, 5, 6, 9, 10, 5, 4, 2, 9, 10. What is the median of his data?

3. Order the integers in each group from least to greatest.
   a. 0, 115, −35, 32, −116, 92
   b. −2, 31, −5, 27, 0, 90

4. Determine each difference.
   a. \( \frac{2}{5} - 1 \frac{1}{2} \)
   b. \( 3 - 1 \frac{1}{3} \)
LESSON 4: You Chose . . . Wisely   •   M5-117

LEARNING GOALS
• Determine whether the mean or median most appropriately represents a typical value in a data set.
• Understand how the distribution of a data set affects the different measures of central tendency and relate the choice of measures of center and variability to the context.
• Determine when to use the interquartile range and the mean absolute deviation to describe the variation of a data set.

WARM UP
Determine each measure for the data given in the stem-and-leaf plot.

Tweets Per Day

| 0 | 2 3 6 6 7 |
| 1 | 0 0 1 5 |
| 2 | 0 |
| 3 | |
| 4 | |
| 5 | |

Key: 2|0 = 20

1. mean
2. median
3. mean absolute deviation

You have learned about different measures of center and different measures of variation. Which of these measures are appropriate to use for data with different characteristics?
Getting Started

Nothing Changes, Nothing Stays the Same

1. Calculate the median and the mean of each data set.
   a. 10 20 30 40 50
   b. 10 20 25 35 40 50
   c. 10 20 30 40 500

2. Create a box-and-whisker plot for each data set.

3. What patterns in the medians and means do you notice?

What can you tell about a data set just by looking at the numbers?
ACTIVITY

Choosing Median or Mean

The dot plot shows the amount of time Ben’s friends spend exercising on weekdays.

Time Spent Exercising Each Weekday

1. Ben says, “The mean will be greater than the median in this data set.” Do you agree with Ben’s statement? Explain your reasoning.

2. Determine the median and mean for the exercise data set. Explain how you determined each.

3. Would the mean or the median be the better measure to describe a typical value in the exercise data? Explain your reasoning.
4. The stem-and-leaf plot shown displays the scores of students on a 100-point math test.

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
</tr>
</tbody>
</table>

Key: $7|1 = 71$ points

**a.** How many students are represented in the data?

**b.** Describe the shape of the distribution of the data.

**c.** Do you think the mean test score is greater than, less than, or about the same as the median score? Explain.

**d.** Determine the median and mean. Identify which measure better represents a typical value in the data. Explain your reasoning.
5. The histogram shown displays the number of hours students spend playing video games each week.

![Histogram showing hours spent playing video games]

a. How many students are represented in the data?

b. Describe the shape of the distribution of the data.

c. Identify which measure—median or mean—would better represent a typical value in the data. Explain your reasoning.
You have learned about three common distributions of data: skewed left, skewed right, and symmetric. You have also learned that the distribution of data can affect the measures of center.

Study the diagrams.

**skewed right**
The mean of a data set is greater than the median when the data is skewed to the right.

**symmetric**
The mean and median are equal when the data is symmetric.

**skewed left**
The mean of a data set is less than the median when the data is skewed to the left.

The median is the best measure of center because the median is not affected by very large data values.

The median is the best measure of center because the median is not affected by very small data values.

The median is not affected by very large or very small data values, but the mean is affected by these large and small values.
1. For each plot shown, first describe the distribution of data. Then, determine whether the mean is less than, greater than, or about equal to the median.

a. Height of students in Room 201

b. Number of text messages sent by 6th graders

c. Rock-Climbing Times of 6th-Grade Students

2. For each part in Question 1, determine whether the median or mean should be used to describe the center of the data.
When a participant takes part in the Special Olympics, they receive a number. The table represents the first 18 people labeled by their participation number and the number of gold medals each participant won.

1. Analyze the data. Calculate the mean and mean absolute deviation, and then interpret the meaning of each in terms of the problem situation.

2. Construct a box-and-whisker plot of the data. Then determine and interpret the IQR.
3. Shelly says that the median and mean absolute deviation should be used to describe the data because the mean absolute deviation is less than the interquartile range. Is Shelly correct? Explain why or why not.

4. Which measure of central tendency and measure of variation should you use to describe the data? Explain your reasoning.

5. What conclusions can you draw about the number of gold medals participants won?
Data were collected from two airlines measuring the difference in the stated departure times and the times the flights actually departed. The average departure time differences were recorded for each month for one year. The results are shown in the two stem-and-leaf plots.

### Difference in Departure Times (minutes)

<table>
<thead>
<tr>
<th>My Air Airlines</th>
<th>Fly High Airlines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stem</td>
<td>Leaf</td>
</tr>
<tr>
<td>0</td>
<td>0 5</td>
</tr>
<tr>
<td>1</td>
<td>1 5 9</td>
</tr>
<tr>
<td>2</td>
<td>0 0 6</td>
</tr>
<tr>
<td>3</td>
<td>3 3 4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>5 = 15 minutes</td>
</tr>
</tbody>
</table>
1. Describe the distribution of each data set.

2. Determine an appropriate measure of central tendency and measure of variation for each data set. Then calculate each measure.

3. What conclusions can you draw from the measure of central tendency and measure of variation you chose?

4. You are scheduling a flight for an important meeting and you must be there on time. Which airline would you schedule with? Explain your reasoning.
TALK the TALK

All Together Now!

For each data set, calculate the median, mean, IQR, and MAD, if possible. Explain which measure of center and which measure of variation best describe the data set.

1. Pencils in Backpacks

<table>
<thead>
<tr>
<th>Number of Pencils</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>X</td>
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<td></td>
</tr>
</tbody>
</table>

2. Ages of U.S. First Ladies (20th Century)

<table>
<thead>
<tr>
<th>Ages</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Key: 6|0 means 60.

3. Prepare a presentation of your analysis of the data from Question 2 to give to the class.
Assignment

Write
In your own words, describe how you would decide whether to use the median or mean to represent the center of a data set.

Remember
When a data set is skewed right, the mean will be greater than the median. When a data set is skewed left, the mean will be less than the median. When a data set is symmetric, the mean and median will be approximately equal.

Practice
Branson Creek Middle School has decided to make fitness a key message to their students in the upcoming school year. As a result, they will be participating in a national fitness program. To participate, they must randomly select 15 students in the 5th grade and record their exercise time each day. The data (in minutes) are shown.

85, 80, 76, 78, 82, 88, 80, 80, 110, 85, 85, 82, 83, 88, 76

1. Construct a dot plot of the data.
2. Describe the distribution of the data.
3. Determine the median and mean of the data. Explain which measure better represents a typical value in the data set.
4. Determine which measure of variation to use to describe the spread of the data. Then calculate this measure.
5. Interpret the measure of variation you calculated.

Stretch
Cecile is applying for a job. She says that it must be a great place to work because it has a really high average salary. Explain to Cecile why this average might be misleading. Provide an example set of data to justify your argument.
Review

1. Complete each table to determine the mean absolute deviation.

a. Complete the table to determine the mean absolute deviation.

<table>
<thead>
<tr>
<th>Data</th>
<th>Mean</th>
<th>Deviation From the Mean</th>
<th>Absolute Value of the Deviation From the Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>65</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mean Absolute Deviation

b. Complete the table to determine the mean absolute deviation.

<table>
<thead>
<tr>
<th>Data</th>
<th>Mean</th>
<th>Deviation From the Mean</th>
<th>Absolute Value of the Deviation From the Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>73</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Mean Absolute Deviation

2. Write the coordinates of each point described. Identify the quadrant in which the point is located.

a. This point is a reflection across the y-axis of the point at (7, 1.5).

b. This point is a reflection across the x-axis of the point at (3, 9).

3. Determine each quotient.

a. \( \frac{3}{5} \div \frac{4}{5} \)

b. \( \frac{7}{8} \div \frac{1}{2} \)
Numerical Summaries of Data Summary

KEY TERMS
- measure of center
- mode
- median
- balance point
- mean
- measures of variation
- range
- quartile
- interquartile range (IQR)
- box-and-whisker plot
- deviation
- absolute deviation
- mean absolute deviation

When you analyze a set of data, you often want to describe it numerically. One way to numerically describe a data set is to use a measure of center. A measure of center tells you how the data values are clustered, or where the center of a graph of the data is located. There are three measures that describe how a data set is centered: the mean, the median, and the mode.

The mode is the data value or values that occur most frequently in a data set. A data set can have more than one mode or no mode. For example, the mode of the data set 12, 6, 12, 26, 4, and 12 is 12.

The median is the middle number in a data set when the values are placed in order from least to greatest or greatest to least. When a data set has an odd number of data values, you can determine which number is exactly in the middle of the data set. If there is an even number of data values, then the median is calculated by adding the two middle numbers and dividing by 2.

For example, the median of the data set 15, 12, 13, 10, 8, and 14 is 12.5.
The third measure of center is based on leveling off or creating fair shares. For example, if you had a stack of two cubes and a stack of six cubes, you can rearrange the stacks to create two equal stacks of four cubes each.

You can also represent quantities on a number line and create a balance point. When you have all the points at the same value, the number line is balanced. The value where the number line is balanced is called the **balance point**.

For example, consider the data set 2, 6.

![Number line with cubes]

The value 2 was moved to the right from 2 to 4. To maintain balance, 6 was moved to the left from 6 to 4. The balance point is 4. The balance point can also be called the mean. The **mean** is the arithmetic average of the numbers in a data set.

For example, determine the mean of the data set: 12, 12, 6, 26, 4, 12.

The mean is calculated by adding all the values in the data set and dividing the sum by the number of values.

**Step 1:**

\[
72 ÷ 6 = 12
\]

You can verify that the mean is 12 because the balance point of the data set is 12.
A measure of variation describes the spread of data values. One measure of variation is the range. The range is the difference between the maximum and minimum values of a data set.

For example, the range of the data 200, 150, 260, 180, 300, 240, and 280 is \(300 - 150 = 150\).

Another set of values that helps to describe variation in a data set is a quartile. When data in a set are arranged in order, quartiles are the numbers that split data into quarters (or fourths). Quartiles are often denoted by the letter \(Q\) followed by a number that indicates which fourth it represents. Since the median is the second quartile, it could be denoted \(Q_2\). The other quartiles are \(Q_1\) and \(Q_3\). The interquartile range, abbreviated IQR, is the difference between the third quartile, \(Q_3\), and the first quartile, \(Q_1\). The IQR indicates the range of the middle 50 percent of the data.

To summarize and describe the spread of the data values, you can use the five-number summary. The five-number summary includes these 5 values from a data set:

- Minimum: the least value in the data set
- \(Q_1\): the first quartile
- Median: the median of the data set
- \(Q_3\): the third quartile
- Maximum: the greatest value in a data set.

For the data set 24, 32, 16, 18, 30, and 20, the minimum is 16, \(Q_1\) is 18, the median is 22, \(Q_3\) is 30, and the maximum is 32.

A box-and-whisker plot, or just box plot, is a graph that displays the five-number summary of a data set.
Box-and-whisker plots can be represented vertically as well as horizontally.

For example, in this box-and-whisker plot, the minimum of the data set is 15, Q1 is 40, the median of the data set is 56, Q3 is 70, and the maximum of the data set is 90.

Another measure of variation that describes the spread of data values is deviation. The deviation of a data value indicates how far that data value is from the mean. To calculate the deviation, subtract the mean from the data value:

\[
\text{Deviation} = \text{data value} - \text{mean}
\]

For example, the mean of the data set 15, 12, 13, 10, 9, and 13 is 12.

The table describes each data point’s deviation from the mean.

<table>
<thead>
<tr>
<th>Data Point</th>
<th>15</th>
<th>12</th>
<th>13</th>
<th>10</th>
<th>9</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deviation from the Mean</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>-3</td>
<td>1</td>
</tr>
</tbody>
</table>

In order to get an idea of the spread of the data values, you can take the absolute value of each deviation and then determine the mean of those absolute values. The absolute value of each deviation is called the absolute deviation. The mean absolute deviation (MAD) is the mean of the absolute deviations.

For example, the mean absolute deviation of the data shown in the table is

\[
\frac{|3| + |0| + |1| + |-2| + |-3| + |1|}{6} = \frac{10}{6},
\]

So, the MAD is about 1.67.
The distribution of data can affect the measures of center.

The median is not affected by very large or very small data values, but the mean is affected by these large and small values. Therefore, the median is the best measure of center when the data is skewed left or right.

For example, the dot plot shows the amount of time Ben’s friends spend exercising on weekdays.

\[
\text{Time Spent Exercising Each Weekday}
\]

The data is skewed right, so the mean is greater than the median. The median for the data set is 60 minutes and the mean is 73.33 minutes. The median is a better measure to describe a typical value in the data.

The measure of central tendency and measure of variation used to best describe a data set depends on the values in the data set and the spread of those values. If you use the median to describe the measure of center, you should use the IQR to describe the measure of variation, and if you use the mean to describe the measure of center, you should use the mean absolute deviation to describe the measure of variation.
absolute deviation
The absolute value of each deviation is called the absolute deviation.

Example
\[ 11 - 12 = -1 \]
\[ \uparrow \quad \uparrow \quad \uparrow \]
data mean deviation
\[ |{-1}| = 1 \]
\[ \uparrow \quad \uparrow \quad \text{absolute deviation} \]

absolute value
The absolute value, or magnitude, of a number is its distance from zero on a number line.

Example
The absolute value of \(-3\) is the same as the absolute value of \(3\) because they are both a distance of \(3\) from zero on a number line.

\[ |{-3}| = |3| \]

Addition Property of Equality
The Addition Property of Equality states that if two values \(a\) and \(b\) are equal, when you add the same value \(c\) to each, the sums are equal.

Examples
\[ 12 = 12 \quad \text{and} \quad 12 + 7 = 12 + 7 \]
If \(a = b\), then \(a + c = b + c\).

additive reasoning
Additive reasoning focuses on the use of addition and subtraction for comparisons.

Example
Vicki is 40 years old and Ben is 10 years old. In 5 years, Vicki will be 45 and Ben will be 15. Vicki will always be 30 years older than Ben. This is additive reasoning.

algebraic expression
An algebraic expression is a mathematical phrase that has at least one variable, and it can contain numbers and operation symbols.

Examples
\[ a \quad 2a + b \quad xy \quad \frac{4}{p} \quad z^2 \]

altitude
The altitude of a figure is the perpendicular distance from a vertex to the line containing the opposite side, represented by a line segment.

Examples
altitude of a parallelogram
altitude of a triangle
altitude of a trapezoid
**balance point**

When you have all the points on a number line at the same value, the number line is balanced. The value where the number line is balanced is called the balance point.

**Example**

Consider the data set: 2, 6.

\[
\begin{align*}
2 + 2 & = 4 & 6 - 2 & = 4 \\
\boxed{2} & \quad \boxed{6}
\end{align*}
\]

The balance point is 4.

**bar graph**

A bar graph displays categorical data using either horizontal or vertical bars on a graph. The height or length of each bar indicates the value for that category.

**Examples**

*Profits from Bake Sale*

Day 1 | Day 2 | Day 3
---|---|---
Profit ($) | 0 | 12 | 20

**bar model**

A bar model uses rectangular bars to represent known and unknown quantities.

**Example**

You can use a bar model to solve the equation \(x + 10 = 15\).

The top bar can be split into two bars, \(x\) and 10. When this split happens in the bottom bar, with one bar containing 10, it shows that \(x\) is the same as 5, so \(x = 5\).

**base**

The base of a power is the factor that is multiplied repeatedly in the power.

**Examples**

\[
\begin{align*}
2^3 & = 2 \times 2 \times 2 = 8 \\
8^0 & = 1
\end{align*}
\]

**benchmark fractions**

Benchmark fractions are common fractions you can use to estimate the value of fractions.

**Example**

The numbers 0, \(\frac{1}{2}\), and 1 are some benchmark fractions.

**benchmark percents**

A benchmark percent is a percent that is commonly used, such as 1%, 5%, 10%, 25%, 50%, and 100%.
**box-and-whisker plot**
A box-and-whisker plot, or just box plot, is a graph that displays the five-number summary of a data set: the median, the upper and lower quartiles (Q1 and Q3), and the minimum and maximum values.

**Example**

Data: 32, 35, 35, 53, 55, 60, 60, 61, 61, 74, 74
Minimum = 32
Q1 = 35
Median = 60
Q3 = 61
Maximum = 74

**clusters**
Clusters are areas of the graph where data are grouped close together.

**Example**

<table>
<thead>
<tr>
<th>Number of Pets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
</tr>
<tr>
<td>X</td>
</tr>
<tr>
<td>X</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

There are clusters of data from 0 to 1 and from 3 to 4.

**coefficient**
A number that is multiplied by a variable in an algebraic expression is called a coefficient.

**Examples**

14x
\( \frac{1}{3}(g) \)

w + 2.5

The coefficient is 1 even though it is not shown.

**common factor**
A common factor is a number that is a factor of two or more numbers.

**Example**

Factors of 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60
Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24
Common factors of 60 and 24: 1, 2, 3, 4, 6, and 12

**Commutative Property of Multiplication**
The Commutative Property of Multiplication states that for any numbers a and b, the product a·b is equal to the product b·a.

**Examples**

\[ \frac{29}{87} = \frac{3}{29} \]
\[ \frac{1}{5} \times \frac{2}{3} = \frac{2}{5} \times \frac{1}{3} \]

GLOSSARY • G-3
complex fraction
A complex fraction is a fraction that has a fraction in either the numerator, the denominator, or both the numerator and denominator.

Examples
\(\frac{\frac{3}{4}}{\frac{1}{2}}, \text{ and } \frac{\frac{1}{4}}{\frac{2}{3}}\) are all complex fractions.

composite solid
A composite solid is made up of more than one geometric solid.

Example

continuous graph
A continuous graph is a graph with no breaks in it.

Examples

convert
To convert a measurement means to change it to an equivalent measurement in different units.

Example
To convert 36 inches to feet, you can multiply:

\[36 \text{ in} \left( \frac{1 \text{ ft}}{12 \text{ in}} \right) = \frac{36 \text{ ft}}{12} = 3 \text{ ft}\]

cube
A cube is a polyhedron that has congruent squares as faces.

Example

D

data
Data are categories, numbers, or observations gathered in response to a statistical question.

Examples
favorite foods of sixth graders, heights of different animals at the zoo

Density Property
The Density Property states that between any two rational numbers there is another rational number.

dependent quantity
The dependent quantity is the quantity that depends on another in a problem situation.

Example
Max just got a new hybrid car that averages 51 miles to the gallon. How far does the car travel on 15 gallons of fuel?

\[\text{number of gallons} \cdot \frac{\text{miles}}{\text{gallon}} = \text{miles traveled}\]

The dependent quantity is the total miles traveled. The number of miles traveled depends on the gallons of fuel.
**dependent variable**

The variable that represents the dependent quantity is called the dependent variable.

**Example**

Max just got a new hybrid car that averages 51 miles to the gallon. How far does the car travel on 15 gallons of fuel?

number of gallons \( \cdot \) miles \( \div \) gallon = miles traveled

\( g \cdot m = t \)

The dependent quantity is the total miles traveled. Since \( t \) represents total miles traveled in the equation, \( t \) is the dependent variable.

---

**deviation**

The deviation of a data value indicates how far that data value is from the mean.

**Example**

deviation = data value − mean

---

**discrete graph**

A discrete graph is a graph of isolated points.

**Examples**

---

**distribution**

The overall shape of a graph is called the distribution of data. A distribution is the way in which the data are spread out.

---

**Distributive Property**

The Distributive Property states that for any numbers \( a, b, \) and \( c, a(b + c) = ab + ac \).

**Examples**

\[
4(2 + 15) = 4 \cdot 2 + 4 \cdot 15 \\
= 8 + 60 \\
= 68
\]

---

**Division Property of Equality**

The Division Property of Equality states that when you divide equal values \( a \) and \( b \) by the same value \( c \) and \( c \neq 0 \), the quotients are equal.

**Examples**

\[
12 = 12 \div 7 = 12 \div 7
\]

If \( a = b \) and \( c \neq 0 \), then \( \frac{a}{c} = \frac{b}{c} \).

---

**dot plot**

A dot plot (sometimes called a line plot) is a data display that shows discrete data on a number line with dots, Xs, or other symbols.

**Example**

---

GLOSSARY • G-5
double number line
A double number line is a model that is made up of two number lines used together to represent the ratio between two quantities.

Example

\[
\begin{array}{c|c|c|c}
\text{Cost ($)} & 0 & 2.50 & 5.00 & 7.50 \\
\text{Number of corn muffins} & 0 & 3 & 6 & 9 \\
\end{array}
\]

equivalent expressions
Two algebraic expressions are equivalent expressions if, when any values are substituted for variables, the results are equal.

Example

\[
(x + 10) + (6x - 5) = 7x + 5 \\
12 + 7 = 14 + 5 \\
19 = 19
\]
equivalent ratios
Equivalent ratios are ratios that represent the same part-to-part or part-to-whole relationship.
evaluate an algebraic expression
To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable.

Example

Evaluate the expression \( \frac{4x + (2^3 - y)}{p} \) for \( x = 2.5 \), \( y = 8 \), and \( p = 2 \).

- First replace the variables with numbers: \( \frac{4(2.5) + (2^3 - 8)}{2} \)
- Then calculate the value of the expression: \( \frac{10 + 0}{2} = \frac{10}{2} = 5 \).
evaluate a numeric expression
To evaluate a numeric expression means to simplify the expression to a single numeric value.

Example

\[
19 - 4 \times 3 \\
19 - 12 \\
7
\]
**experiment**

An experiment is one method of collecting data in which a researcher imposes a condition and observes the results.

**Example**

A researcher conducts an experiment to investigate if 6th graders perform better on an assessment if they read a textbook or watch a video about the material. The researcher randomly assigns half the students to read the text and half the students to watch the video. All students would be given the same assessment and the scores of the students in the two groups would be compared.

---

**exponent**

The exponent of the power is the number of times the base is used as a factor.

**Examples**

\[2^3 = 2 \times 2 \times 2 \]

\[8^4 = 8 \times 8 \times 8 \times 8 \]

---

**face**

A face is one of the polygons that makes up a polyhedron.

**Example**

---

**frequency**

A frequency is the number of times an item or number occurs in a data set.

**Example**

<table>
<thead>
<tr>
<th>Number Rolled</th>
<th>Tally</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2 2 2</td>
<td>7</td>
</tr>
</tbody>
</table>

The number 2 was rolled 7 times, so its frequency was 7.

---

**gaps**

Gaps are areas of the graph where there are no data.

**Example**

There are gaps between 1 and 3 and between 4 and 7.

---

**geometric solid**

A geometric solid is a bounded three-dimensional geometric figure.

**Example**
**graph of an inequality**

The graph of an inequality in one variable is the set of all points on a number line that make the inequality true.

**Example**

\[ x \leq 3 \]

**grouped frequency table**

A grouped frequency table is a table used to organize data according to how many times data values with a given range of values occur.

**Example**

<table>
<thead>
<tr>
<th>Floor Intervals</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10–20</td>
<td>8</td>
</tr>
<tr>
<td>20–30</td>
<td>27</td>
</tr>
<tr>
<td>30–40</td>
<td>16</td>
</tr>
<tr>
<td>40–50</td>
<td>5</td>
</tr>
<tr>
<td>50–60</td>
<td>4</td>
</tr>
</tbody>
</table>

**greatest common factor (GCF)**

The greatest common factor, or GCF, is the largest factor two or more numbers have in common.

**Example**

- factors of 16: 1, 2, 4, 8, 16
- factors of 12: 1, 2, 3, 4, 6, 12
- common factors: 1, 2, 4
- greatest common factor: 4

**histogram**

A histogram is a graphical way to display quantitative or numerical data using vertical bars. The width of a bar represents an interval of data and is often referred to as a bin. The height of the bar indicates the frequency, or the number of data values included in any given bin.

**Example**

![Histogram](image)

**Identity Property of Addition**

The Identity Property of Addition states that the sum of any number and 0 is the number.

**Examples**

\[ 6 \times 0 = 6 \quad \frac{3}{4} + 0 = \frac{3}{4} \]
\[ 5^2 + 0 = 5^2 \quad 0.125 + 0 = 0.125 \]

**Identity Property of Multiplication**

The Identity Property of Multiplication states that the product of any number and 1 is the number.

**Examples**

\[ 6 \times 1 = 6 \quad \frac{3}{4} \times 1 = \frac{3}{4} \]
\[ 5^2 \times 1 = 5^2 \quad 0.125(1) = 0.125 \]
**independent quantity**

The independent quantity is the quantity the dependent quantity depends on.

**Example**

Max just got a new hybrid car that averages 51 miles to the gallon. How far does the car travel on 15 gallons of fuel?

number of gallons \( \times \) \( \frac{\text{miles}}{\text{gallon}} \) = miles traveled

The independent quantity is the number of gallons. The other quantity (miles traveled) is dependent upon this quantity.

**independent variable**

The variable that represents the independent quantity is called the independent variable.

**Example**

Max just got a new hybrid car that averages 51 miles to the gallon. How far does the car travel on 15 gallons of fuel?

number of gallons \( \times \) \( \frac{\text{miles}}{\text{gallon}} \) = miles traveled

\( g \times m = t \)

The independent quantity is the number of gallons. Since \( g \) represents the number of gallons in the equation, \( g \) is the independent variable.

**infinity**

Infinity, represented by the symbol \( \infty \), means a quantity with no end or bound.

**Example**

Negative infinity

\( \cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \rightarrow \infty \)

Positive infinity

\( 0, 1, 2, 3, 4, 5 \rightarrow \infty \)

**integers**

Integers are the set of whole numbers with their opposites.

**Example**

The set of integers can be represented as \( \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \)

**interquartile range (IQR)**

The interquartile range, abbreviated IQR, is the difference between the third quartile, Q3, and the first quartile, Q1. The IQR indicates the range of the middle 50 percent of the data.

**Example**

\( \begin{array}{c}
35 = Q1 \\
IQR = 61 - 35 = 26 \\
Q3 = 61 \\
\end{array} \)

**inverse operations**

Inverse operations are pairs of operations that reverse the effects of each other.

**Examples**

Addition and subtraction are inverse operations: \( 351 + 25 - 25 = 351 \).

Multiplication and division are inverse operations: \( 351 \times 25 \div 25 = 351 \).

**kite**

A kite is a quadrilateral with two pairs of consecutive congruent sides where opposite sides are not congruent.

**Example**

A kite is shown with sides AB = 1 cm, AD = 1 cm, BC = 3 cm, and DC = 3 cm.
least common multiple (LCM)

The least common multiple, or LCM, is the smallest multiple (other than zero) that two or more numbers have in common.

**Example**
multiples of 60: 60, 120, 180, 240, 300, 360, 420, 480 . . .
multiples of 24: 24, 48, 72, 96, 120, 144, 168, 192, 216, 240 . . .
some common multiples of 60 and 24: 120, 240 . . .
least common multiple of 60 and 24: 120

like terms

In an algebraic expression, like terms are two or more terms that have the same variable raised to the same power.

**Examples**
like terms

\[4x + 3p + x + 2 = 5x + 3p + 2\]

no like terms

\[m + m^2 - x + x^3\]

line segment

A line segment is a portion of a line that includes two points and all the points between those two points.

**Example**

Line segment \(AB\) is shown.

linear relationship

When a set of points graphed on a coordinate plane forms a straight line, a linear relationship exists.

**Example**

The points graphed show a linear relationship.

![Graph of a linear relationship](image)

literal equation

A literal equation is an equation in which the variables represent specific measures.

**Examples**

\[A = lw \quad A = \frac{1}{2} bh \quad d = rt\]

mean

The mean is the arithmetic average of the numbers in a data set.

**Example**

\[
\text{Number of Pets} \\
x \quad x \quad x \quad x \quad x \quad x \\
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\
\text{Mean} = \frac{0 + 0 + 1 + 1 + 1 + 3 + 3 + 5}{9} = \frac{15}{9} = 1\frac{2}{3} \text{ pets}
\]
**mean absolute deviation**

The mean absolute deviation is the average or mean of the absolute deviations.

**measure of center**

A measure of center tells you how the data values are clustered, or where the “center” of a graph of the data is located.

**Examples**

Mean, median, and mode are each a measure of center for data.

**measure of variation**

A measure of variation describes the spread of data values.

**Example**

Range is a measure of variation for data.

**median**

The median is the middle number in a data set when the values are placed in order from least to greatest or greatest to least.

**Example**

The mode is the value or values that occur most frequently in a data set.

**Example**

The mode of the data is 1.

**multiple**

A multiple is the product of a given whole number and another whole number.

**Example**

Multiples of 10:

\[ 10, 20, 30, 40, 50, \ldots \]

**Multiplication Property of Equality**

The Multiplication Property of Equality states that if two values \( a \) and \( b \) are equal, when you multiply each by the same value \( c \), the products are equal.

**Examples**

\[ 12 = 12 \text{ and } 12(7) = 12(7) \]

If \( a = b \), then \( ac = bc \).

**multiplicative inverse**

The multiplicative inverse of a number \( \frac{a}{b} \) is the number \( \frac{b}{a} \), where \( a \) and \( b \) are nonzero numbers. The product of any nonzero number and its multiplicative inverse is 1.

**Examples**

The multiplicative inverse of \( \frac{3}{7} \) is \( \frac{7}{3} \): \( \frac{3}{7} \times \frac{7}{3} = \frac{21}{21} = 1 \)
The multiplicative inverse of 5 is \( \frac{1}{5} \): \( \frac{5}{1} \times \frac{1}{5} = \frac{5}{5} = 1 \)
Multiplicative Inverse Property

The Multiplicative Inverse Property states: \( \frac{a}{b} \cdot \frac{b}{a} = 1 \), where \( a \) and \( b \) are nonzero numbers.

**Examples**

\[
\frac{3}{7} \times \frac{7}{3} = \frac{21}{21} = 1 \\
\frac{5}{1} \times \frac{1}{5} = \frac{5}{5} = 1 
\]

multiplicative reasoning

Multiplicative reasoning focuses on the use of multiplication and division.

**Example**

Vicki is 40 years old and Ben is 10 years old. Vicki is 4 times as old as Ben. In 5 years, Vicki will be 3 times as old as Ben.

This is multiplicative reasoning.

negative numbers

The values to the left of zero on a number line are called negative numbers.

**Example**

![Negative numbers number line](image)

net

A net is a two-dimensional representation of a three-dimensional geometric figure.

**Example**

A net of a cube is shown.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

numeric expression

A numeric expression is a mathematical phrase that contains numbers and operations.

**Example**

\[ 5 \times 4 - 9 \]

observational study

An observational study is one method of collecting data in which a researcher collects data by observing the variable of interest.

**Example**

A researcher is interested in whether or not more men or women prefer a certain store. The researcher observes the number of men and women who visit the store over a number of hours and compares the values of the two groups.

one-step equation

A one-step equation is an equation that can be solved using only one operation.

Order of Operations

The Order of Operations is a set of rules that ensures the same result every time an expression is evaluated.

**Example**

\[
44 + (6 - 5) - 2 \times 75 \div 5 \\
44 + 1 - 2 \times 75 \div 5 \rightarrow \text{Parentheses} \\
44 + 1 - 2 \times 75 \div 5 \rightarrow \text{Exponents} \\
44 + 1 - 150 \div 5 \rightarrow \text{Multiplication and Division} \\
44 + 1 - 30 \rightarrow \text{(from left to right)} \\
45 - 30 \rightarrow \text{Addition and Subtraction} \\
15 \rightarrow \text{(from left to right)}
\]
outliers
Outliers are data values that lie a large distance from the other data in a graph. Outliers usually accompany gaps in data.

Example

<table>
<thead>
<tr>
<th>Number of Pets</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
</tr>
<tr>
<td>X</td>
</tr>
<tr>
<td>X X</td>
</tr>
<tr>
<td>X X</td>
</tr>
<tr>
<td>X X</td>
</tr>
<tr>
<td>X</td>
</tr>
</tbody>
</table>

The value 7 is an outlier.

percent
A percent is a part-to-whole ratio where the whole is equal to 100. Percent is another name for hundredths. The percent symbol “%” means “per 100,” or “out of 100.”

perfect cube
A perfect cube is the cube of a whole number.

Example

64 is a perfect cube: $4 \times 4 \times 4 = 64$

perfect square
A perfect square is the square of an integer.

Examples

9 is a perfect square: $3 \times 3 = 9$

25 is a perfect square: $5 \times 5 = 25$

point
A point is a location in space. A point has no size or shape, but it is often represented by using a dot and is named by a capital letter.

Examples

Points A and B are shown.

A •

B
**polygon**

A polygon is a closed figure formed by three or more line segments.

**Examples**

A trapezoid is a polygon.

A pentagon is a polygon.

A circle is NOT a polygon.

**polyhedron**

A polyhedron is a three-dimensional solid figure that has polygons as faces.

**Example**

A cube is a polyhedron. It has six square faces.

**population**

A population is an entire set of items from which data are collected.

**Example**

If you wanted to determine the average height of the students at your school, the number of students at the school would be the population.

**positive rational number**

A positive rational number is a number that can be written in the form \( \frac{a}{b} \), where \( a \) and \( b \) are both whole numbers greater than 0.

**Examples**

\( 0.75 = \frac{75}{100} \), where \( a = 75 \) and \( b = 100 \)

\( 6 = \frac{6}{1} \), where \( a = 6 \) and \( b = 1 \)

\( \frac{9}{11} \), where \( a = 9 \) and \( b = 11 \)

**power**

A power has two elements: the base and the exponent.

**Example**

base \( \rightarrow 6^2 \) exponent

\( 6^2 \)

**proportion**

A proportion is an equation that states that two ratios are equal.

**Example**

\( \frac{1}{2} = \frac{4.5}{9} \)

**pyramid**

A pyramid is a polyhedron with one base and the same number of triangular faces as there are sides of the base.

**Example**
quadrants

The $x$- and $y$-axes divide the coordinate plane into four regions called quadrants. These quadrants are numbered with Roman numerals from one (I) to four (IV), starting in the upper right-hand quadrant and moving counterclockwise.

Example

![Diagram of quadrant plane]

quantitative data

Quantitative data, or numerical data, are data for which each piece of data can be placed on a numerical scale and compared.

Examples

The zoo has 4 lions, 3 tigers, and 6 bears.
In 2006, Los Angeles had a population of about 3,849,378. In the same year, Atlanta had a population of about 429,500.

quartiles

Quartiles are a set of values that describe variation in a data set. When data in a set are arranged in order, quartiles are the numbers that split data into quarters (or fourths).

Example

<table>
<thead>
<tr>
<th>Data: 32, 35, 35, 53, 55, 60, 60, 61, 61, 74, 74</th>
</tr>
</thead>
<tbody>
<tr>
<td>first quartile (Q1) third quartile (Q3)</td>
</tr>
<tr>
<td>32</td>
</tr>
<tr>
<td>35</td>
</tr>
<tr>
<td>35</td>
</tr>
<tr>
<td>53</td>
</tr>
<tr>
<td>55</td>
</tr>
<tr>
<td>60</td>
</tr>
<tr>
<td>60</td>
</tr>
<tr>
<td>61</td>
</tr>
<tr>
<td>61</td>
</tr>
<tr>
<td>74</td>
</tr>
<tr>
<td>second quartile/median (Q2)</td>
</tr>
</tbody>
</table>

range

The range is the difference between the maximum and minimum values of a data set.

Example

<table>
<thead>
<tr>
<th>Number of Pets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 0, 1, 1, 1, 1, 3, 3, 5</td>
</tr>
<tr>
<td>5 – 0 = 5</td>
</tr>
</tbody>
</table>

The range of the data is 5.

rate

A rate is a ratio that compares two quantities that are measured in different units.

Example

The speed of 60 miles in two hours is a rate:

$$\frac{60 \text{ mi}}{2 \text{ h}} = \frac{30 \text{ mi}}{1 \text{ h}}$$

ratio

A ratio is a comparison of two quantities that uses division.

Examples

The ratio of stars to circles is $\frac{3}{2}$, or 3:2, or 3 to 2.
The ratio of circles to stars is $\frac{2}{3}$, or 2:3, or 2 to 3.

rational numbers

Rational numbers are the set of numbers that can be written as $\frac{a}{b}$, where $a$ and $b$ are integers and $b \neq 0$.

Examples

$-4, \frac{1}{2}, \frac{2}{3}, 0.67$, and $\frac{22}{7}$ are examples of rational numbers.
reciprocal

The reciprocal of a number is also known as the multiplicative inverse of the number. (See multiplicative inverse.)

**Examples**

The reciprocal of $\frac{3}{7}$ is $\frac{7}{3}$. $\frac{3}{7} \times \frac{7}{3} = \frac{21}{21} = 1$

The reciprocal of 5 is $\frac{1}{5}$. $\frac{1}{5} \times \frac{1}{5} = \frac{5}{5} = 1$

**Reflexive Property of Equality**

The Reflexive Property of Equality says that when both sides of an equation look exactly the same, their values are equal.

**Examples**

$7 = 7$

$a = a$

**relatively prime**

Two numbers that do not have any common factors other than 1 are called relatively prime.

**Examples**

Positive whole number pairs that have a difference of 1 (4 and 5, 10 and 11, 15 and 16) are always relatively prime.

**right rectangular prism**

A right rectangular prism is a polyhedron with three pairs of congruent and parallel rectangular faces.

**Example**

![Right Rectangular Prism Diagram]

**sample**

A sample is a selection from a population.

**Example**

If you wanted to determine the average height of the students in your school, you could choose a certain number of students and measure their heights. The heights of the students in this group would be your sample.

**scaling down**

Scaling down means to divide both parts of the ratio by the same factor greater than 1, or multiply both parts of the ratio by the same factor less than 1.

**Example**

$\frac{3}{6} = \frac{1}{2}$. $\frac{3}{6} \div 3 = \frac{1}{2}$

**scaling up**

Scaling up means to multiply both parts of a ratio by the same factor greater than 1.

**Example**

$\frac{1}{3} \times 3 = \frac{3}{3}$. $\frac{1}{3} \times 3 = \frac{3}{3}$
skewed left distribution

In a skewed left distribution of data the peak of the data is to the right side of the graph. There are only a few data points to the left side of the graph.

Example

![Skewed Left Distribution Diagram]

skewed right distribution

In a skewed right distribution of data the peak of the data is to the left side of the graph. There are only a few data points to the right side of the graph.

Example

![Skewed Right Distribution Diagram]

slant height

A slant height of a pyramid is the distance measured along a triangular face from the vertex of the pyramid to the midpoint, or center, of one of the edges of the base.

Example

![Slant Height in a Pyramid Diagram]

treatment

A treatment is an intervention or treatment applied to a group in an experiment to measure its effect.

Example

![Treatment Diagram]

summary

A summary is a brief account or statement that is intended to convey the essence of something.

Example

![Summary Diagram]

solution

A solution to an equation is any value for a variable that makes the equation true.

Example

The solution to the equation $2x + 4 = 8$ is $x = 2$.

solution set of an inequality

The set of all points that make an inequality true is the solution set of the inequality.

Examples

$x \geq 7$

The solution set for $x \geq 7$ is all the numbers greater than or equal to 7.

$x < 7$

The solution set for $x < 7$ is all the numbers less than 7.

statistical process

The statistical process has four components:

• Formulating a statistical question.
• Collecting appropriate data.
• Analyzing the data graphically and numerically.
• Interpreting the results of the analysis.

statistical question

A statistical question is a question that anticipates an answer based on data that vary.

Example

“What sport is the most popular in your school?” is a statistical question because it anticipates that the answers will vary since not everyone at your school is likely to have the same favorite sport.

“How many students are in Chess Club?” is NOT a statistical question because there is only one answer to the question.
stem-and-leaf plot

A stem-and-leaf plot is a graphical method used to represent ordered numerical data. Once the data are ordered, the stem and leaves are determined. Typically, the stem is all the digits in a number except the rightmost digit, which is the leaf.

**Example**

**Books Read in Mr. Brown’s Class**

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3, 6</td>
<td>0, 1, 5</td>
<td>9, 9</td>
<td>0, 0, 0</td>
<td></td>
</tr>
</tbody>
</table>

Key: 1 | 0 = 10.

Subtraction Property of Equality

The Subtraction Property of Equality states that when you subtract the same value \( c \) from equal values \( a \) and \( b \), the differences are equal.

**Examples**

\[ 12 = 12 \text{ and } 12 - 7 = 12 - 7 \]

If \( a = b \), then \( a - c = b - c \).

surface area

The surface area of a polyhedron is the total area of all its two-dimensional faces.

**Example**

The surface area of a unit cube is 6 square units. The cube has 6 faces and the area of each face is 1 square unit.

survey

A survey is one method of collecting data in which people are asked one or more questions.

**Example**

A restaurant may ask its customers to complete a survey with the following question:

On a scale of 1–10, with 1 meaning “poor” and 10 meaning “excellent,” how would you rate the food you ate?

\[ \square 1 \square 2 \square 3 \square 4 \square 5 \square 6 \square 7 \square 8 \square 9 \square 10 \]

symmetric distribution

In a symmetric distribution of data the left and right halves of the graph are mirror images of each other. The peak is in the middle because there are many data values in the center.

**Example**

Symmetric Property of Equality

The Symmetric Property of Equality states that if \( a = b \), then \( b = a \).

**Example**

\( x = 3 \text{ is the same as } 3 = x \).
tape diagram
A tape diagram illustrates number relationships by using rectangles to represent ratio parts.

Example
A bakery sells packs of muffins in the ratio of 3 blueberry muffins : 2 pumpkin muffins : 1 bran muffin. The tape diagram represents the ratio of each type of muffin.

term
A term of an algebraic expression is a number, variable, or product of numbers and variables.

Example
The expression has four terms.
\[3y + 5xy + \frac{1}{2}x + 6\]

trailing zeros
Trailing zeros are a sequence of 0s in a decimal representation of a number, after which no non-zero digits follow.

Example
9,500

trapezoid
A trapezoid is a quadrilateral with two bases that are parallel to each other, often labeled \(b_1\) and \(b_2\).

Example
Quadrilateral ABCD is a trapezoid. Side BC is parallel to side AD.

unit rate
A unit rate is a comparison of two different measurements in which the numerator or denominator has a value of one unit.

Example
The speed 60 miles in 2 hours can be written as a unit rate:
\[
\frac{60 \text{ mi}}{2 \text{ h}} = \frac{30 \text{ mi}}{1 \text{ h}}.
\]
The unit rate is 30 miles per hour.

variability
In statistics, variability means that the value of the attribute being studied can change from one person or thing to another.

variable
A variable is a letter or symbol that is used to represent a number.

Examples
\[3x = 81\]
**vertex**

A vertex of a polyhedron is a point at which three or more of its edges meet.

**Example**

![Diagram of a polyhedron with vertices labeled]

**volume**

Volume is the amount of space occupied by an object. Volume is measured in cubic units.

**Zero Property of Multiplication**

The Zero Property of Multiplication states that the product of any number and 0 is 0.

**Examples**

- $6 \times 0 = 0$
- $\frac{3}{4} \times 0 = 0$
- $5^2 \times 0 = 0$
- $0.125 \times 0 = 0$
Symbols
|| (absolute value), M4-25, M4-49
\approx (approximately equal to), M2-169
… (ellipsis), M4-50
\equiv (equals), M1-108, M3-112, M4-17–M4-18
( ) (grouping symbol), M3-14–M3-16, M3-18
[ ] (grouping symbol), M3-16
\infty (infinity), M4-9
3 (multiplication), M1-44
2 (negative sign), M4-9, M4-10
% (percent), M2-18
1 (plus sign), M4-9
\neq (less than), M1-108, M3-112, M4-17–M4-18
\leq (less than or equal to), M1-108, M3-112
\leq, \geq, \neq (inequality symbols), M1-108, M3-112, M4-17–M4-18
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